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Probability Theory

Clark–Ocone type formula for non-semimartingales with finite quadratic variation

Formule de Clark–Ocone generalisée pour des non-semimartingales à variation quadratique finie

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A R T I C L E   I N F O

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A B S T R A C T

We provide a suitable framework for the concept of finite quadratic variation for processes with values in a separable Banach space B using the language of stochastic calculus via regularizations, introduced in the case $B = \mathbb{R}$ by the second author and P. Vallois. To a real continuous process X we associate the Banach-valued process $X(\cdot)$, called *window* process, which describes the evolution of X taking into account a memory $\tau > 0$. The natural state space for $X(\cdot)$ is the Banach space of continuous functions on $[-\tau, 0]$. If X is a real finite quadratic variation process, an appropriated Itô formula is presented, from which we derive a generalized Clark–Ocone formula for non-semimartingales having the same quadratic variation as Brownian motion. The representation is based on solutions of an infinite-dimensional PDE.

R É S U M É

Nous présentons un cadre adéquat pour le concept de variation quadratique finie lorsque le processus de référence est à valeurs dans un espace de Banach séparable B. Le langage utilisé est celui de l'intégrale via régularisations introduit dans le cas réel par le second auteur et P. Vallois. À un processus réel continu X, nous associons le processus $X(\cdot)$, appelé processus *fenêtre*, qui à l'instant t, garde en mémoire le passé jusqu'à $t - \tau$. L'espace naturel d'évolution pour $X(\cdot)$ est l'espace de Banach B des fonctions continues définies sur $[-\tau, 0]$. Si X est un processus réel à variation quadratique finie, nous énonçons une formule d'Itô appropriée de laquelle nous déduisons une formule de Clark–Ocone relative à des non-semimartingales réelles ayant la même variation quadratique que le mouvement brownien. La représentation est basée sur des solutions d'une EDP infini-dimensionnelle.

Version française abrégée

Dans cette Note nous développons un calcul stochastique via régularisation de type progressif (*forward*) lorsque le processus intégrateur $\mathbb{X}$ est à valeurs dans un espace de Banach séparable B. Ceci est basé sur une notion sophistiquée de

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variation quadratique que nous appellerons χ-variation quadratique, où le symbole χ correspond à un sous-espace χ du dual du produit tensoriel projectif $B \hat{\otimes}_\pi B$. Le calcul via régularisation a été introduit lorsque $B = \mathbb{R}$ en 1991 par F. Russo et P. Vallois et depuis il a été étudié par de nombreux auteurs qui ont fait avancer la théorie et ont produit plusieurs applications. Le lecteur peut consulter [7] pour une revue incluant une liste assez complète de références. Dans ce contexte, les auteurs introduisent une notion de covariation entre deux processus réels X et Y, notée $[X, Y]$ qui généralise le crochet droit usuel lorsque X et Y sont des semimartingales. Un vecteur de processus $\underline{X} = (X^1, \dots, X^n)$ est dit admettre tous ses crochets mutuels si $[X^i, X^j]$ existe pour tous entiers $1 \leqslant i, j \leqslant n$.

Lorsque $B = \mathbb{R}^n$, $\mathbb{X}$ possède une χ-variation quadratique avec $\chi = (B \hat{\otimes}_\pi B)^*$ si et seulement si $\mathbb{X}$ admet tous ses crochets mutuels. On peut voir qu'un semimartingale $\mathbb{X}$ à valeurs dans un espace de Banach au sense de [6] admet une χ-variation quadratique avec $\chi = (B \hat{\otimes}_\pi B)^*$. Dans ce travail nous traçons une ébauche du calcul stochastique via la formule d'Itô énoncée au Théorème 4.1. Une attention spéciale est consacrée au cas où B est l'espace $C([-\tau, 0])$ des fonctions continues définies sur $[-\tau, 0]$, pour un certain $\tau > 0$, qui est typiquement un espace de Banach non-réflexif et à une formule de Clark–Ocone généralisée. Motivés par des applications liées à la couverture d'options dépendant de toute la trajectoire, nous discutons une formule de type Clark–Ocone visant à décomposer une classe significative de v.a. h dépendant de la trajectoire d'un processus X dont la variation quadratique vaut $[X]_t = t$. Cette formule généralise des résultats inclus dans [8] visant à déterminer des formules de valorisation et de couverture d'options vanille où asiatique dans un modèle de prix d'actif ayant la même variation quadratique que le modèle de Black–Scholes. Si le bruit dans un environnement stochastique est modélisé par la dérivée d'un mouvement brownien W, le théorème de représentation des martingales et la formule classique de Clark–Ocone sont deux outils fondamentaux de calcul. Le Théorème 6.1 et les considérations à la fin de la Section 6 montrent que dans une certaine mesure une formule de type Clark–Ocone reste valable lorsque la loi du processus soujacent n'est plus la mesure de Wiener mais le processus conserve la même variation quadratique que W.

1. Introduction and notations

In the whole paper $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ fulfilling the usual conditions, B will be a separable Banach space and $\mathbb{X}$ a B-valued process. If K is a compact set, $\mathcal{M}(K)$ will denote the space of Borel (signed) measures on K. $C([-\tau, 0])$ will denote the space of continuous functions defined on $[-\tau, 0]$ whose topological dual space is $\mathcal{M}([-\tau, 0])$. W will always denote an $(\mathcal{F}_t)$-real Brownian motion. Let $T > 0$ be a fixed maturity time. All the processes $X = (X_t)_{t \in [0,T]}$ are prolongated by continuity for $t \notin [0, T]$ setting $X_t = X_0$ for $t \leqslant 0$ and $X_t = X_T$ for $t \geqslant T$.

We first recall the basic concepts of forward integral and covariation and some one-dimensional results concerning calculus via regularization, a fairly complete survey on the subject being [7]. For simplicity, all the considered integrator processes will be continuous.

Let X (respectively Y) be a continuous (resp. locally integrable) process. The *forward integral of Y with respect to X* (resp. the *covariation of X and Y*), whenever it exists, is defined as

$$\int_0^t Y_s \,\mathrm{d}^- X_s := \lim_{\epsilon \to 0^+} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} \,\mathrm{d}s \quad \left(\text{resp. } [X, Y]_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) \,\mathrm{d}s\right), \tag{1}$$

in probability for all $t \in [0, T]$ provided that the limiting process admits a continuous version (resp. in the ucp sense with respect to t). If $\int_0^t Y_s \,\mathrm{d}^- X_s$ exists for any $0 \leqslant t < T$; $\int_0^T Y_s \,\mathrm{d}^- X_s$ will symbolize the *improper forward integral* defined by $\lim_{t \to T} \int_0^t Y_s \,\mathrm{d}^- X_s$, whenever it exists in probability. If $[X, X]$ exists then X is said to be a *finite quadratic variation* process. $[X, X]$ will also be denoted by $[X]$ and it will be called *quadratic variation of* X. If $[X] = 0$, then X is said to be a *zero quadratic variation process*. If $\mathbb{X} = (X^1, \dots, X^n)$ is a vector of continuous processes we say that it has all its *mutual covariations* (brackets) if $[X^i, X^j]$ exists for any $1 \leqslant i, j \leqslant n$.

When X is a (continuous) semimartingale (resp. Brownian motion) and Y is an adapted cadlag process (resp. such that $\int_0^T Y_s^2 \,\mathrm{d}s < \infty$ a.s.), the integral $\int_0^\cdot Y_s \,\mathrm{d}^- X_s$ exists and coincides with classical Itô's integral $\int_0^\cdot Y_s \,\mathrm{d}X_s$, see Proposition 6 in [7]. Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate those integrals when Y is anticipating or X is not a semimartingale. If X, Y are $(\mathcal{F}_t)$-semimartingales then $[X, Y]$ coincides with the classical bracket $\langle X, Y \rangle$, see Corollary 2 in [7]. Finite quadratic variation processes will play a central role in this Note: this class includes of course all $(\mathcal{F}_t)$-semimartingales. However that class is much richer. Typical examples of finite quadratic variation processes are $(\mathcal{F}_t)$-Dirichlet processes. D is called $(\mathcal{F}_t)$-Dirichlet process if it admits a decomposition $D = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process. It holds in that case $[D] = [M]$. This class of processes generalizes the semimartingales since a locally bounded variation process has zero quadratic variation. A slight generalization of that notion is the concept of weak Dirichlet process, which was introduced in [5]. In several circumstances such a process has again finite quadratic variation.

One central object of this work will be the generalization to infinite-dimensional-valued processes of the stochastic integral via regularization, see Definition 2.1. A stochastic calculus for Banach (or Hilbert) valued martingales was considered for instance in the monographies [2,6,4].

We introduce now a particular Banach-valued process. Given $0 < \tau \leqslant T$ and a real continuous process X, we will call *window process* associated with X, the $C([-\tau,0])$-valued process denoted by $X(\cdot)$ defined as $X(\cdot) = (X_t(\cdot))_{t\in[0,T]} = \{X_t(u) := X_{t+u};\, u \in [-\tau,0],\, t \in [0,T]\}$. The window process $W(\cdot)$ associated with the classical Brownian motion W will be called *window Brownian motion*. We observe that $W(\cdot)$ is not a $B = C([-\tau,0])$-valued semimartingale in the sense of [6], see Definition 1 in Section 10.8. In fact this would in particular imply that ${}_{B^*}\langle \mu, W(\cdot)\rangle_B$ is a real semimartingale for any $\mu \in B^*$. This is not the case since setting $\mu = \delta_0 + \delta_{-\tau/2}$, the process $Y_t := {}_{\mathcal{M}([-\tau,0])}\langle \mu, W_t(\cdot)\rangle_{C([-\tau,0])} = \int_{-\tau}^{0} W_t(u)\,\mathrm{d}\mu(u) = W_t + W_{t-\frac{\tau}{2}}$ which is not a real semimartingale. In fact its canonical filtration is the filtration $(\mathcal{F}_t)$ associated with W. Taking into account Corollary 3.14 of [1] Y is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part W. By uniqueness of the decomposition of a weak Dirichlet process (see Proposition 16 of [7]) Y cannot be an $(\mathcal{F}_t)$-semimartingale.

Motivated by the necessity of an Itô formula available also for $B = C([-\tau,0])$-valued processes, we introduce a quadratic variation concept which depends on a subspace χ of the dual of the tensor square of B, equipped with the projective topology, denoted by $(B \hat{\otimes}_\pi B)^*$, see Definition 3.1. We recall the fundamental identification $(B \hat{\otimes}_\pi B)^* \cong \mathcal{B}(B \times B)$, which denotes the space of $\mathbb{R}$-valued bounded bilinear forms on $B \times B$. An Itô formula for processes admitting a χ-quadratic variation is given in Theorem 4.1. After formulating a theory for B-valued processes with general B, in Sections 5 and 6 we fix the attention on window processes setting $B = C([-\tau,0])$. Section 5, in particular Proposition 5.3, is devoted to the evaluation of χ-quadratic variation for windows associated with real finite quadratic variation processes. Suppose that X is a real process such that $[X]_t = t$. In Section 6 we give a representation result for a random variable $h := H(X_T(\cdot))$ where $H : C([-T,0]) \to \mathbb{R}$ is continuous. That is of the type $h = H_0 + \int_0^T \xi_s\,\mathrm{d}^- X_s$, $H_0 \in \mathbb{R}$ and ξ adapted process where the integral is considered as the forward integral defined in (1). More precisely h will appear as $u(T, X_T(\cdot))$ where $u \in C^{1,2}([0,T[\times C([-T,0]);\mathbb{R}) \cap C^0([0,T] \times C([-T,0]);\mathbb{R})$ solves an infinite-dimensional partial differential equation of type (5). Moreover we will get $H_0 = u(0, X_0(\cdot))$ and $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$ where $D^{\delta_0} u(t,\eta) := Du(t,\eta)(\{0\})$; Du denotes the Fréchet derivative with respect to $\eta \in C([-T,0]$ so $Du(t,\eta)$ is a signed measure.

Symbol $\mathscr{C}([0,T])$ denotes the linear space of continuous real processes equipped with the ucp (uniformly convergence in probability) topology, B^* will be the topological dual of the Banach space B. We introduce now some subspaces of measures that we will frequently use. Symbol $\mathcal{D}_0([-\tau,0])$ (resp. $\mathcal{D}_{0,0}([-\tau,0]^2)$), shortly $\mathcal{D}_{0,0}$ (resp. $\mathcal{D}_{0,0}$), will denote the one-dimensional Hilbert space generated by the Dirac measure concentrated at 0 (resp. at $(0,0)$), i.e. $\mathcal{D}_0([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0]);$ s.t. $\mu(\mathrm{d}x) = \lambda\delta_0(\mathrm{d}x)$ with $\lambda \in \mathbb{R}\}$ (resp. $\mathcal{D}_{0,0}([-\tau,0]^2) := \{\mu \in \mathcal{M}([-\tau,0]^2);$ s.t. $\mu(\mathrm{d}x,\mathrm{d}y) = \lambda\delta_0(\mathrm{d}x)\delta_0(\mathrm{d}y)$ with $\lambda \in \mathbb{R}\}$). Symbol $Diag([-\tau,0]^2)$, shortly $Diag$, will denote the subset of $\mathcal{M}([-\tau,0]^2)$ defined as $\{\mu \in \mathcal{M}([-\tau,0]^2)$ s.t. $\mu(\mathrm{d}x,\mathrm{d}y) = g(x)\delta_y(\mathrm{d}x)\,\mathrm{d}y;\; g \in L^\infty([-\tau,0])\}$. $Diag([-\tau,0]^2)$, equipped with the norm $\|\mu\|_{Diag([-\tau,0]^2)} = \|g\|_\infty$, is a Banach space.

2. Forward integrals in Banach spaces

In this section we introduce an infinite-dimensional stochastic integral via regularization. In this construction there are two main difficulties. The integrator is generally not a semimartingale or the integrand may be anticipative; B is a general separable, not necessarily reflexive, Banach space.

Definition 2.1. Let $(\mathbb{X}_t)_{t\in[0,T]}$ (respectively $(\mathbb{Y}_t)_{t\in[0,T]}$) be a B-valued (respectively a B^*-valued) stochastic process. We suppose $\mathbb{X}$ to be continuous and $\mathbb{Y}$ to be strongly measurable (in the Bochner sense) such that $\int_0^T \|\mathbb{Y}_s\|_{B^*}\,\mathrm{d}s < +\infty$ a.s.

For every fixed $t \in [0,T]$ we define the *definite forward integral of* $\mathbb{Y}$ *with respect to* $\mathbb{X}$ denoted by $\int_0^t {}_{B^*}\langle \mathbb{Y}_s, \mathrm{d}^-\mathbb{X}_s\rangle_B$ as the following limit in probability: $\int_0^t {}_{B^*}\langle \mathbb{Y}_s, \mathrm{d}^-\mathbb{X}_s\rangle_B := \lim_{\epsilon\to 0} \int_0^t {}_{B^*}\langle \mathbb{Y}_s, \frac{\mathbb{X}_{s+\epsilon} - \mathbb{X}_s}{\epsilon}\rangle_B\,\mathrm{d}s$. We say that the *forward stochastic integral of* $\mathbb{Y}$ *with respect to* $\mathbb{X}$ exists if the process $(\int_0^t {}_{B^*}\langle \mathbb{Y}_s, \mathrm{d}^-\mathbb{X}_s\rangle_B)_{t\in[0,T]}$ admits a continuous version. In the sequel indices B and B^* will often be omitted.

3. Chi-quadratic variation

A closed linear subspace χ of $(B \hat{\otimes}_\pi B)^*$, endowed with its own norm, such that $\|\cdot\|_{(B\hat{\otimes}_\pi B)^*} \leqslant \mathrm{const}\|\cdot\|_\chi$ will be called a *Chi-subspace (of* $(B \hat{\otimes}_\pi B)^*$*)*. Let χ be a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$, $\mathbb{X}$ be a B-valued stochastic process and $\epsilon > 0$. We denote by $[\mathbb{X}]^\epsilon$, the application $[\mathbb{X}]^\epsilon : \chi \to \mathscr{C}([0,T])$ defined by $\phi \mapsto (\int_0^t {}_\chi\langle \phi, \frac{J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2)}{\epsilon}\rangle_{\chi^*}\,\mathrm{d}s)_{t\in[0,T]}$ where $J : B \hat{\otimes}_\pi B \to (B \hat{\otimes}_\pi B)^{**}$ denotes the canonical injection between a space and its bidual.

We recall that $\chi \subset (B \hat{\otimes}_\pi B)^*$ implies $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$. As indicated, ${}_\chi\langle\cdot,\cdot\rangle_{\chi^*}$ denotes the duality between the space χ and its dual χ^*; in fact, by assumption, ϕ is an element of χ and element $J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2)$ naturally belongs to $(B \hat{\otimes}_\pi B)^{**} \subset \chi^*$. The real function $s \to \langle \phi, J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2)\rangle$ is integrable since $|\langle \phi, J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2)\rangle| \leqslant \mathrm{const}\|\phi\|_\chi \|\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\|_B^2$. With a slight abuse of notation, in the sequel, the application J will be omitted. The tensor product $(X_{s+\epsilon} - X_s)\otimes^2$ has to be considered as the element $J((\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2)$ which belongs to χ^*. We give now the definition of the χ-quadratic variation of a B-valued stochastic process $\mathbb{X}$.

Definition 3.1. Let χ be a separable Chi-subspace of $(B \hat{\otimes}_\pi B)^*$ and $\mathbb{X}$ a B-valued stochastic process. We say that $\mathbb{X}$ *admits a* χ*-quadratic variation* if the following assumptions are fulfilled.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leqslant 1} \left| {}_\chi\left\langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s)\otimes^2}{\epsilon_{n_k}} \right\rangle_{\chi^*} \right| \mathrm{d}s < +\infty \quad \text{a.s.}$$

H2 It exists an application denoted by $[\mathbb{X}] : \chi \to \mathscr{C}([0,T])$, such that $[\mathbb{X}]^\epsilon(\phi) \underset{\epsilon\to 0_+}{\overset{ucp}{\to}} [\mathbb{X}](\phi)$ for all $\phi \in \mathcal{S}$, where $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$.

We formulate a technical proposition which is stated in Corollary 4.38 in [3].

Proposition 3.2. *Suppose that $\mathbb{X}$ admits a χ-quadratic variation. Then convergence in* H2 *holds for any $\phi \in \chi$ and $[\mathbb{X}]$ is a linear continuous application. In particular $[\mathbb{X}]$ does not depend on $\mathcal{S}$. Moreover there exists a χ^*-valued measurable process $(\widetilde{[\mathbb{X}]})_{0\leqslant t\leqslant T}$, cadlag and with bounded variation on $[0,T]$ such that $\widetilde{[\mathbb{X}]}_t(\cdot)(\phi) = [\mathbb{X}](\phi)(\cdot, t)$ a.s. for any $t \in [0,T]$ and $\phi \in \chi$.*

Its proof is based on Banach–Steinhaus and separability arguments. The existence of $\widetilde{[\mathbb{X}]}$ guarantees that $[\mathbb{X}]$ admits a proper version which allows to consider it as a pathwise integral.

Definition 3.3. When $\mathbb{X}$ admits a χ-quadratic variation, the χ^*-valued measurable process $(\widetilde{[\mathbb{X}]})_{0\leqslant t\leqslant T}$ appearing in Proposition 3.2, is called *χ-quadratic variation* of $\mathbb{X}$. Sometimes, with a slight abuse of notation, even $[\mathbb{X}]$ will be called χ-quadratic variation and it will be confused with $\widetilde{[\mathbb{X}]}$. We say that a continuous B-valued process $\mathbb{X}$ admits *global quadratic variation* if it admits a χ-quadratic variation with $\chi = (B \hat{\otimes}_\pi B)^*$. In particular $\widetilde{[\mathbb{X}]}$ takes values "a priori" in $(B \hat{\otimes}_\pi B)^{**}$.

The natural generalization of quadratic variation for a B-valued process is a $(B \hat{\otimes}_\pi B)$-valued process, called the *tensor quadratic variation*, as it was introduced by [4] and [6]. This is operational for stochastic calculus when a real-valued process, called *real quadratic variation* exists. Unfortunately, the real quadratic variation does not exist in several contexts. For instance, the window Brownian motion $W(\cdot)$, which is our fundamental example, does not admit it, see Remark 5.2. The tensor quadratic variation is related to a strong convergence in $B \hat{\otimes}_\pi B$ while our concept of global quadratic variation is related to a weak star convergence in its bidual. If $\mathbb{X}$ admits a real and tensor quadratic variation then it admits a global quadratic variation, see Section 6.3 in [3] for details. When B is the finite-dimensional space $\mathbb{R}^n$, $\mathbb{X}$ admits a real and tensor quadratic variation and if and only if $\mathbb{X}$ admits a global quadratic variation. In that case previous properties are also equivalent to the existence of all the mutual brackets in the sense of [7].

4. Itô's formula

The classical Itô formulae for stochastic integrators $\mathbb{X}$ with values in an infinite-dimensional space appear in Section 4.5 of [2] for the Hilbert separable case and in Section 3.7 in [6], see also [4], as far as the Banach case is concerned; they involve processes admitting a tensor quadratic variation. We state now an Itô formula in the general separable Banach space which do not necessarily have a tensor quadratic variation but they have rather a χ-quadratic variation, where χ is some Chi-subspace where the second order Fréchet derivative lives. This type of formula is well suited for $C([-\tau,0])$-valued integrators as for instance window processes; this will be developed in Sections 5 and 6. In the sequel if $F : [0,T] \times B \to \mathbb{R}$ then (if it exists) DF (resp. D^2F) stands for the first (resp. second) order Fréchet derivative with respect to the B variable.

Theorem 4.1. *Let B be a separable Banach space, χ be a Chi-subspace of $(B \hat{\otimes}_\pi B)^*$ and $\mathbb{X}$ a B-valued continuous process admitting a χ-quadratic variation. Let $F : [0,T] \times B \to \mathbb{R}$ of class $C^{1,2}$ Fréchet such that $D^2F : [0,T] \times B \to \chi \subset (B \hat{\otimes}_\pi B)^*$ is continuous with respect to χ. Then the forward integral $\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), \mathrm{d}^-\mathbb{X}_s\rangle_B$, $t \in [0,T]$, exists and the following formula holds*

$$F(t,\mathbb{X}_t) = F(0,\mathbb{X}_0) + \int_0^t \partial_t F(s,\mathbb{X}_s)\,\mathrm{d}s + \int_0^t {}_{B^*}\big\langle DF(s,\mathbb{X}_s), \mathrm{d}^-\mathbb{X}_s\big\rangle_B + \frac{1}{2}\int_0^t {}_\chi\big\langle D^2F(s,\mathbb{X}_s), \mathrm{d}\widetilde{[\mathbb{X}]}_s\big\rangle_{\chi^*} \quad \textit{a.s.} \tag{2}$$

5. Evaluation of χ-quadratic variations for window processes

From this section we fix B as the Banach space $C([-\tau,0])$. In this section we give some examples of Chi-subspaces and then we give some evaluations of χ-quadratic variations for window processes $\mathbb{X} = X(\cdot)$. For illustration of possible applications of Itô formula (2), consider the following functions. Let $H : B \to \mathbb{R}$ and $\eta \in B$ defined by (a) $H(\eta) = f(\eta(0))$, $f \in C^2(\mathbb{R})$; (b) $H(\eta) = (\int_{-\tau}^0 \eta(s)\,\mathrm{d}s)^2$ and (c) $H(\eta) = \int_{-\tau}^0 \eta^2(s)\,\mathrm{d}s$. Those functions are of class $C^2(B)$; computing the second order Fréchet derivative $D^2H : B \to (B \hat{\otimes}_\pi B)^*$ we obtain the following: (a) $D^2_{\mathrm{d}x\,\mathrm{d}y}H(\eta) = f''(\eta(0))\delta_0(\mathrm{d}x)\delta_0(\mathrm{d}y)$;

(b) $D^2H(\eta) = 2\mathbb{1}_{[-\tau,0]^2}$ and (c) $D^2_{\mathrm{d}x\,\mathrm{d}y}H(\eta) = 2\delta_x(\mathrm{d}y)\,\mathrm{d}x$. In all those examples, $D^2H(\eta)$ lives in a particular Chi-subspace χ. Respectively we have $D^2H : B \to \chi$ continuously with (a) $\chi = \mathcal{D}_{0,0}([-\tau,0]^2)$; (b) $\chi = L^2([-\tau,0]^2)$ and (c) $\chi = Diag([-\tau,0]^2)$. Other examples of Chi-subspaces are $\mathcal{M}([-\tau,0]^2)$ and its subspace $\chi^0([-\tau,0]^2)$, (shortly χ^0), defined by $\chi^0([-\tau,0]^2) := (\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0]))\hat{\otimes}_h^2$, where $\hat{\otimes}_h$ stands for the Hilbert tensor product. The latter one will intervene in Theorem 6.1 in relation with the generalized Clark–Ocone formula. We evaluate now some χ-quadratic variations of window processes.

Proposition 5.1. *Let X be a real-valued process with Hölder continuous paths of parameter $\gamma > 1/2$. Then $X(\cdot)$ admits a zero global quadratic variation.*

Examples of real processes with Hölder continuous paths of parameter $\gamma > 1/2$ are fractional Brownian motion B^H with $H > 1/2$ or a bifractional Brownian motion $B^{H,K}$ with $HK > 1/2$.

Remark 5.2. The window Brownian motion $W(\cdot)$ does not admit a global (and therefore not a tensor) quadratic variation because condition H1 is not verified. In fact it is possible to show that $\int_0^T \frac{1}{\epsilon}\|W_{u+\epsilon}(\cdot) - W_u(\cdot)\|_B^2\,\mathrm{d}u \geqslant TA^2(\tilde{\epsilon})\ln(1/\tilde{\epsilon})$ where $\tilde{\epsilon} = 2\epsilon/T$ and $(A(\epsilon))$ is a family of non negative r.v. such that $\lim_{\epsilon\to 0} A(\epsilon) = 1$ a.s.

Proposition 5.3. *Let X be a real continuous process with finite quadratic variation $[X]$ and $0 < \tau \leqslant T$. The following properties hold true.*

(1) *$X(\cdot)$ admits zero $L^2([-\tau,0]^2)$-quadratic variation.*
(2) *$X(\cdot)$ admits a $\mathcal{D}_{0,0}([-\tau,0]^2)$-quadratic variation given by $[X(\cdot)](\mu) = \mu(\{0,0\})[X]$, $\forall\mu \in \mathcal{D}_{0,0}([-\tau,0]^2)$.*
(3) *$X(\cdot)$ admits a $\chi^0([-\tau,0]^2)$-quadratic variation which equals $[X(\cdot)](\mu) = \mu(\{0,0\})[X]$, $\forall\mu \in \chi^0([-\tau,0]^2)$.*
(4) *$X(\cdot)$ admits a Diag-quadratic variation given by $\mu \mapsto [X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X]_{t-x}\,\mathrm{d}x$, $t \in [0,T]$, where μ is a generic element in $Diag([-\tau,0]^2)$ of type $\mu(\mathrm{d}x,\mathrm{d}y) = g(x)\delta_y(\mathrm{d}x)\,\mathrm{d}y$, with associated g in $L^\infty([-\tau,0])$.*

We remark that in the treated cases, the quadratic variation $[X]$ of the real finite quadratic variation process X insures the existence of (and completely determines) the χ-quadratic variation. For example if X is a real finite quadratic variation process such that $[X]_t = t$, then $X(\cdot)$ has the same χ-quadratic variation as the window Brownian motion for the χ mentioned in the above proposition.

6. A generalized Clark–Ocone formula

In this section we will consider $\tau = T$ and we recall that $B = C([-T,0])$. Let X be a real stochastic process such that $X_0 = 0$ and $[X]_t = t$. Let $H : C([-T,0]) \to \mathbb{R}$ be a Borel functional; we aim at representing the random variable $h = H(X_T(\cdot))$. The main task will consist in looking for classes of functionals H for which there is $H_0 \in \mathbb{R}$ and a predictable process ξ with respect to the canonical filtration of X such that h admits the representation $h = H_0 + \int_0^T \xi_s\,\mathrm{d}^- X_s$. Moreover we look for an explicit expression for H_0 and ξ. As a consequence of Itô's formula (2) for path dependent functionals of the process we will observe that, in those cases, it is possible to find a function u which solves an infinite-dimensional PDE and which gives at the same time the representation result giving explicit expressions for H_0 and ξ. One possible representation is the following.

Theorem 6.1. *Let $H : C([-T,0]) \to \mathbb{R}$ be a Borel functional. Let $u \in C^{1,2}([0,T[\times C([-T,0])) \cap C^0([0,T]\times C([-T,0]))$ such that $x \mapsto D_x^{ac}u(t,\eta)$ has bounded variation, for any $t \in [0,T]$, $\eta \in C([-T,0])$ and $D^{ac}u(t,\eta)$ is the absolute continuous part of measure $Du(t,\eta)$. We suppose moreover that $(t,\eta) \mapsto D^2u(t,\eta)$ takes values in $\chi^0([-T,0]^2)$ and it is continuous. Suppose that u is a solution of*

$$\begin{cases} \partial_t u(t,\eta) + \displaystyle\int_{]-t,0]} D^{ac}u(t,\eta)\,\mathrm{d}\eta + \frac{1}{2}D^2u(t,\eta)\big(\{0,0\}\big) = 0, \\ u(T,\eta) = H(\eta), \end{cases} \tag{3}$$

where the integral $\int_{]-t,0]} D^{ac}u(t,\eta)\,\mathrm{d}\eta$ has to be understood via an integration by parts as follows: $\int_{]-t,0]} D^{ac}u(t,\eta)\,\mathrm{d}\eta = D^{ac}u(0,\eta)\eta(0) - D^{ac}u(-t,\eta)\eta(-t) - \int_{]-t,0]}\eta(x)D^{ac}_{\mathrm{d}x}u(t,\eta)$. Then the random variable $h := H(X_T(\cdot))$ admits the following representation

$$h = u\big(0, X_0(\cdot)\big) + \int_0^T D^{\delta_0}u\big(t, X_t(\cdot)\big)\,\mathrm{d}^- X_t. \tag{4}$$

Sections 9.8 and 9.9 in [3] provide different reasonable conditions on $H : C([-T,0]) \to \mathbb{R}$ such that there is a function u solving PDE (3) in general situations, i.e. fulfilling the hypotheses of Theorem 6.1. When $H : C([-T,0]) \subset L^2([-T,0]) \to \mathbb{R}$ is $C^3(L^2([-T,0]))$ such that $D^2 H \in L^2([-T,0]^2)$ with some other minor technical conditions, Theorem 9.41 in [3] furnishes explicit solutions to (3). Another case for which it is possible to do the same is given by Proposition 9.53 in [3], where h depends (not necessarily smoothly) on a finite number of Wiener integrals of the type $\int_0^T \varphi(s)\,\mathrm{d}^- X_s$ and $\varphi \in C^2(\mathbb{R})$.

In relation to Theorem 6.1 we observe that only pathwise considerations intervene and there is no need to suppose that the law of X is Wiener measure. Since $H(\eta) = u(T,\eta)$, we observe that H is automatically continuous by hypothesis $u \in C^0([0,T] \times C([-T,0]))$.

Let us suppose $X = W$. Making use of probabilistic technology, (4) holds in some cases even if H is not continuous and $h \notin L^1(\Omega)$; we refer to Section 9.6 in [3] for this type of results. If $\int_0^T \xi_s^2\,\mathrm{d}s < +\infty$ a.s., then the forward integral $\int_0^T \xi_t\,\mathrm{d}^- W_t$ coincides with the Itô integral $\int_0^T \xi_t\,\mathrm{d}W_t$. If the r.v. $h = H(W_T(\cdot))$ belongs to $\mathbb{D}^{1,2}$, by uniqueness of the martingale representation theorem and point 2, we have $H_0 = \mathbb{E}[h]$ and $\xi_t = \mathbb{E}[D_t^m h|\mathcal{F}_t]$, where D^m is the Malliavin gradient; this agrees with Clark–Ocone formula.

If X is not a Brownian motion, in general $H_0 \neq \mathbb{E}[h]$ since $\mathbb{E}[\int_0^T \xi_t\,\mathrm{d}^- X_t]$ does not generally vanish. In fact $\mathbb{E}[h]$ will specifically depend on the unknown law of X.

In Chapter 9 in [3] we enlarge the discussion presented in Theorem 6.1. We can give examples where $u : [0,T] \times C([-T,0]) \to \mathbb{R}$ of class $C^{1,2}([0,T[\times C([-T,0]);\mathbb{R}) \cap C^0([0,T] \times C([-T,0]);\mathbb{R})$ with $D^2 u \in \chi_0$ such that (4) holds and u solves an infinite-dimensional PDE of the type

$$\begin{cases} \partial_t u(t,\eta) + \text{“}\displaystyle\int_{]-t,0]} D^{ac} u(t,\eta)\,\mathrm{d}\eta\text{”} + \dfrac{1}{2}\langle D^2 u(t,\eta), \mathbb{1}_D\rangle = 0, \\ u(T,\eta) = H(\eta), \end{cases} \tag{5}$$

where $\mathbb{1}_D(x,y) := \begin{cases} 1 & \text{if } x=y,\ x,y\in[-T,0] \\ 0 & \text{otherwise} \end{cases}$. The integral “$\int_{]-t,0]} D^{ac} u(t,\eta)\,\mathrm{d}\eta$” has to be suitably defined and term $\langle D^2 u(t,\eta), \mathbb{1}_D\rangle$ indicates the evaluation of the second order derivative on the diagonal of the square $[-T,0]^2$.

We observe that solution of (3) are also solutions of (5) since $\langle D^2 u(t,\eta), \mathbb{1}_D\rangle = D^2 u(t,\eta)(\{0,0\})$ because $D^2 u$ takes values in χ^0.

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