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Model reduction of semiaffinely parameterized partial differential equations by two-level affine approximation

Réduction de modèle pour des équations aux dérivées partielles paramétrisées semi-affinement par une approximation affine à deux niveaux

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ABSTRACT

We propose an improvement to the reduced basis method for parametric partial differential equations. An assumption of affine parameterization leads to an efficient offline–online decomposition when the problem is solved for many different parametric configurations. We consider an advection–diffusion problem, where the diffusive term is non-affinely parameterized and treated with a two-level affine approximation given by the empirical interpolation method. The offline stage and a posteriori error estimation is performed using the coarse-level approximation, while the fine-level approximation is used to perform a correction iteration that reduces the actual error of the reduced basis approximation while keeping the same certified error bounds.

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RÉSUMÉ

On propose une amélioration de la méthode des bases réduites pour des équations aux dérivées partielles paramétriques. On utilise l'hypothèse de paramétrisation affine pour obtenir un problème ayant des formes bilinéaires indépendantes des paramètres et des fonctions scalaires qui dépendent des paramètres. Ceci mène à une décomposition offline–online (qui est) plus efficace lorsque le problème est résolu pour différentes configurations des paramètres. Toutefois, dans le cas général, la condition de paramétrisation affine n'est pas satisfaite. On considère un problème d'advection–diffusion où la matrice des coefficients d'advection est paramétrisée de manière affine et où on traite le terme diffusif non-affine avec une approximation affine à deux niveaux obtenue avec une méthode d'interpolation empirique. La partie offline est effectuée en utilisant une approximation affine grossière alors qu'une approximation affine plus fine est utilisée pour accomplir une itération de correction.

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On considère une équation paramétrique d'advection–diffusion : soit $\boldsymbol{\mu}$ un vecteur de paramètres, trouver $u(\boldsymbol{\mu}) \in X(\Omega)$ tel que (1) soit vérifié pour tout $v \in X(\Omega)$, où $\Omega \subset \mathbb{R}^d$ est un domaine de Lipschitz borné et $X(\Omega) := H^1_{\Gamma_d}(\Omega)$ est l'espace des fonctions H^1 qui s'annulent sur le bord correspondant aux conditions de Dirichlet $\Gamma_d \subset \partial\Omega$ avec la norme usuelle de Sobolev $\|\cdot\|_{H^1}$. Les formes bilinéaires paramétriques sont définies par (2) où le champ de vitesse $\boldsymbol{w}(\boldsymbol{x})$ est à divergence nulle et, pour simplifier, ne dépend pas des paramètres, et la forme linéaire paramétrique est définie par $\mathcal{F}(v;\boldsymbol{\mu}) \triangleq \int_\Omega f(\boldsymbol{x},\boldsymbol{\mu})v(\boldsymbol{x})\,\mathrm{d}\Omega$. On s'intéresse au modèle de réduction de (1) en utilisant la méthode des bases réduites [5,6]. De travaux précédents sur ce sujet sont présentés dans [7,9]. Dans l'Annexe, on présente brièvement les détails de l'approximation par les bases réduites pour l'équation d'advection–diffusion.

Par la suite, on fait les hypothèses suivantes : (i) Le vecteur des paramètres $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$ est de dimension finie et le domaine des valeurs admissibles des paramètres $\mathcal{D}$ est borné ; (ii) La fonction matricielle $\nu(\boldsymbol{x},\boldsymbol{\mu})$ est positive et uniformément définie positive pour tout $\boldsymbol{x} \in \Omega$ et $\boldsymbol{\mu} \in \mathcal{D}$; (iii) La fonction matricielle $\chi(\boldsymbol{x},\boldsymbol{\mu})$ et le membre de droite $f(\boldsymbol{x};\boldsymbol{\mu})$ peuvent être décomposés de manière affine par (3) pour des entiers Q_b, Q_f, des fonctions scalaires lisses Θ_b^q, Θ_f^q, des fonctions matricielles χ^q et des fonctions scalaires f^q ; (iv) Il existe deux niveaux d'approximations affines (fin et grossier) de la fonction matricielle non-affine $\nu(\boldsymbol{x},\boldsymbol{\mu})$ par (4) et (5) où ν_{coarse} et ν_{fine} sont toutes les deux symétriques et définies positives, et les termes restants satisfont (6) pour chaque $\boldsymbol{\mu} \in \mathcal{D}$ pour certaines tolérances $0 < \delta^{\text{tol}}_{\text{fine}} \ll \delta^{\text{tol}}_{\text{coarse}}$. Ces approximations peuvent être obtenues en utilisant l'interpolation empirique (EIM) introduite dans [1].

Comme la partie advective de nos équations est paramétrisée de manière affine alors que la partie de diffusion ne l'est pas, on dit que l'équation est *paramétrisée de manière semi-affine*. En replacant le coefficient de la matrice de diffusion non-affine ν par son approximation fine ν_{fine} dans (1), on obtient le problème affin : trouver $u(\boldsymbol{\mu}) \in X(\Omega)$ par (7) pour tout $v \in X(\Omega)$, où la forme bilinéaire est définie en replacant ν avec ν_{fine} dans (2). La méthode des bases réduites peut alors être appliquée pour réaliser un schéma efficace de réduction de modèle. En choisissant $\delta^{\text{tol}}_{\text{coarse}}$ assez bas, la solution de ce problème affine est une approximation de la solution de (1), mais typiquement, on trouve que le coût de la partie offline croît considérablement lorsque le nombre de termes dans l'expansion Q_{fine}, croît. La solution de (7) est alors approximée par la séquence de problè suivants. Dans un premier temps, trouver $u^0(\boldsymbol{\mu}) \in X(\Omega)$ par (P0) pour tout $v \in X(\Omega)$. Alors, pour chaque $k = 1, 2, \ldots$ trouver $u^k(\boldsymbol{\mu}) \in X(\Omega)$ par (Pk) pour tout $v \in X(\Omega)$. Il faudrait montrer que la suite $\{u^k\}_{k=1}^\infty$ converge vers la solution de (7). Le schéma des bases réduites peut alors être écrit pour les équations (P0) et (Pk) afin d'obtenir une itération de point fixe dans l'espace réduit qui prend la solution des bases réduites grossière et utilise l'information issue de l'approximation plus fine pour améliorer la qualité de l'approximation.

1. Introduction

We consider a parametric advection–diffusion equation: for a given vector of parameters $\boldsymbol{\mu}$, find $u(\boldsymbol{\mu}) \in X(\Omega)$ s.t.

$$\mathcal{A}(u, v; \boldsymbol{\mu}) + \mathcal{B}(u, v; \boldsymbol{\mu}) = \mathcal{F}(v; \boldsymbol{\mu}) \tag{1}$$

for all $v \in X(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and $X(\Omega) := H^1_{\Gamma_d}(\Omega)$ is the space of H^1-functions that vanish on the Dirichlet part of the boundary $\Gamma_d \subset \partial\Omega$ with the usual Sobolev norm $\|\cdot\|_{H^1}$. The parametric bilinear forms are defined as

$$\mathcal{A}(u, v; \boldsymbol{\mu}) \triangleq \varepsilon \int_\Omega \nu(\boldsymbol{x}, \boldsymbol{\mu})\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\Omega, \qquad \mathcal{B}(u, v; \boldsymbol{\mu}) \triangleq \int_\Omega v(\boldsymbol{x})\boldsymbol{w}(\boldsymbol{x}) \cdot \chi(\boldsymbol{x}, \boldsymbol{\mu})\nabla u(\boldsymbol{x})\,\mathrm{d}\Omega \tag{2}$$

where the advective velocity field $\boldsymbol{w}(\boldsymbol{x})$ is divergence-free and for simplicity does not depend on the parameter, and the parametric linear form is defined as $\mathcal{F}(v; \boldsymbol{\mu}) \triangleq \int_\Omega f(\boldsymbol{x}, \boldsymbol{\mu})v(\boldsymbol{x})\,\mathrm{d}\Omega$. We are interested in parametric model reduction of (1) using the reduced basis method [5,6]. An important part of the reduced model is providing certified a posteriori error bounds between an underlying finite element approximation and its reduced basis approximation. Previous works on this topic include [7,9]. In Appendix A we briefly present the details of reduced basis approximation for the advection–diffusion equation.

We make the following assumptions: (i) The parameter vector $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^P$ is finite-dimensional and the admissible parameter range $\mathcal{D}$ is bounded; (ii) The matrix function $\nu(\boldsymbol{x}, \boldsymbol{\mu}) \in C^0(\mathcal{D}; L^\infty(\Omega; \mathbb{R}^{d\times d}))$ is symmetric and uniformly positive definite for all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{\mu} \in \mathcal{D}$; (iii) The matrix function $\chi(\boldsymbol{x}, \boldsymbol{\mu}) \in C^0(\mathcal{D}; L^\infty(\Omega; \mathbb{R}^{d\times d}))$ and the right-hand side $f(\boldsymbol{x}; \boldsymbol{\mu}) \in C^0(\mathcal{D}; L^2(\Omega))$ can be decomposed affinely into

$$\chi(\boldsymbol{x}, \boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\chi^q(\boldsymbol{x}), \qquad f(\boldsymbol{x}, \boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_q^f(\boldsymbol{\mu}) f^q(\boldsymbol{x}) \tag{3}$$

for some suitable integers Q_b, Q_f and smooth, computable scalar functions Θ_b^q, Θ_f^q and matrix functions χ^q and scalar functions f^q; (iv) There exist two levels of affine approximations (fine and coarse) of the non-affine matrix function $\nu(\boldsymbol{x}, \boldsymbol{\mu})$ s.t.

$$\nu(\boldsymbol{x},\boldsymbol{\mu}) = \sum_{q=1}^{Q_{\text{fine}}} \Theta^q_{\text{fine}}(\boldsymbol{\mu})\nu^q_{\text{fine}}(\boldsymbol{x}) + \varepsilon_{\text{fine}}(\boldsymbol{x},\boldsymbol{\mu}) = \nu_{\text{fine}}(\boldsymbol{x},\boldsymbol{\mu}) + \varepsilon_{\text{fine}}(\boldsymbol{x},\boldsymbol{\mu}) \tag{4}$$

and

$$\nu(\boldsymbol{x},\boldsymbol{\mu}) = \sum_{q=1}^{Q_{\text{coarse}}} \Theta^q_{\text{coarse}}(\boldsymbol{\mu})\nu^q_{\text{coarse}}(\boldsymbol{x}) + \varepsilon_{\text{coarse}}(\boldsymbol{x},\boldsymbol{\mu}) = \nu_{\text{coarse}}(\boldsymbol{x},\boldsymbol{\mu}) + \varepsilon_{\text{coarse}}(\boldsymbol{x},\boldsymbol{\mu}) \tag{5}$$

where ν_{coarse} and ν_{fine} are both symmetric and uniformly positive definite, and the remainder terms satisfy

$$\left\|\varepsilon_{\text{fine}}(\cdot,\boldsymbol{\mu})\right\|_{L^\infty} \leqslant \delta^{\text{tol}}_{\text{fine}}, \qquad \left\|\varepsilon_{\text{coarse}}(\cdot,\boldsymbol{\mu})\right\|_{L^\infty} < \delta^{\text{tol}}_{\text{coarse}} \tag{6}$$

for every $\boldsymbol{\mu} \in \mathcal{D}$ for some prescribed tolerances $0 < \delta^{\text{tol}}_{\text{fine}} \ll \delta^{\text{tol}}_{\text{coarse}}$. These approximations can be obtained by using the empirical interpolation method (EIM) introduced in [1].

Because the advective part of our equations is affinely parameterized while the diffusive part is not, we say that the equation is *semiaffinely parameterized*. By replacing the non-affine diffusive coefficient matrix ν with its fine-level approximation ν_{fine} in (1) we obtain the affine problem to find $u(\boldsymbol{\mu}) \in X(\Omega)$ s.t.

$$\mathcal{A}_{\text{fine}}(u,v;\boldsymbol{\mu}) + \mathcal{B}(u,v;\boldsymbol{\mu}) = \mathcal{F}(v;\boldsymbol{\mu}) \tag{7}$$

for all $v \in X(\Omega)$, where the affinely parameterized bilinear form is defined by replacing ν with ν_{fine} in (2).

If the reduced basis method is applied directly to (7), the cost of computing the coercivity lower bound α_{LB} is high since the workload scales polynomially in Q_{fine}, i.e. certifying the reduced solution is expensive. The solution of (7) is therefore approximated by the following sequence of problems. In the first step, find $u^0(\boldsymbol{\mu}) \in X(\Omega)$ s.t.

$$\mathcal{A}_{\text{coarse}}\big(u^0,v;\boldsymbol{\mu}\big) + \mathcal{B}\big(u^0,v;\boldsymbol{\mu}\big) = \mathcal{F}(v;\boldsymbol{\mu}) \tag{P0}$$

for all $v \in X(\Omega)$. Then for each $k = 1, 2, \ldots$ find $u^k(\boldsymbol{\mu}) \in X(\Omega)$ s.t.

$$\mathcal{A}_{\text{coarse}}\big(u^k,v;\boldsymbol{\mu}\big) + \mathcal{B}\big(u^k,v;\boldsymbol{\mu}\big) = \mathcal{F}(v;\boldsymbol{\mu}) + \mathcal{A}_{\text{coarse}}\big(u^{k-1},v;\boldsymbol{\mu}\big) - \mathcal{A}_{\text{fine}}\big(u^{k-1},v;\boldsymbol{\mu}\big) \tag{Pk}$$

for all $v \in X(\Omega)$. We shall show that the sequence $\{u^k\}_{k=1}^\infty$ converges to the solution of (7) when $\delta^{\text{tol}}_{\text{coarse}}$ is sufficiently small. The cost of the offline stage for the problems (P0) and (Pk) is much smaller since $Q_{\text{coarse}} \ll Q_{\text{fine}}$. Once the coarse-level reduced system with certified error bounds has been obtained, from the convergence of the fixed point algorithm we also get a certificate for the fine-level problem with the same error bounds. Thus the reduced basis method has been split into two levels of approximation detail: the coarse-level tolerance $\delta^{\text{tol}}_{\text{coarse}}$ controls the certification of the error bounds and cost of the offline stage, while the $\delta^{\text{tol}}_{\text{fine}}$ controls the actual error of the reduced basis solution. The user of the reduced model can then balance between the computational time spent in preparing the reduced model and the size of the certified error bounds without sacrificing the accuracy of the reduced model.

2. Convergence proof for the correction iteration

Theorem 2.1. *Assume that the coarse-level approximation ν_{coarse} has tolerance at most $\delta^{\text{tol}}_{\text{coarse}} < \frac{\alpha_{\text{LB}}}{2\varepsilon d}$, where $\alpha_{\text{LB}} > 0$ is the uniform lower bound for the coercivity constant of the coarse-level problem, i.e.*

$$0 < \alpha_{\text{LB}} \leqslant \alpha(\boldsymbol{\mu}) = \inf_{v \in X(\Omega)} \frac{\mathcal{A}_{\text{coarse}}(v,v;\boldsymbol{\mu}) + \mathcal{B}(v,v;\boldsymbol{\mu})}{\|v\|^2_{H^1}} \quad \textit{for all } v \in X(\Omega),\ \boldsymbol{\mu} \in \mathcal{D}. \tag{8}$$

Then the sequence $\{u^k\}_{k=0}^\infty$ given by (P0) *and* (Pk) *converges to a solution of* (7).

Proof. By subtracting (P(k+1)) and (Pk) from each other implies that the difference $w := u^{k+1} - u^k$ satisfies $\mathcal{A}_{\text{coarse}}(w,v) + \mathcal{B}(w,v) = \mathcal{G}^k(v)$, where $\mathcal{G}^k(v) := \varepsilon \int_\Omega (\nu_{\text{coarse}} - \nu_{\text{fine}})\nabla(u^k - u^{k-1}) \cdot \nabla v \, \mathrm{d}\Omega$. The standard stability result due to the Lax–Milgram theorem gives $\|u^{k+1} - u^k\|_{H^1} \leqslant \alpha^{-1}(\boldsymbol{\mu})\|\mathcal{G}^k\|_{H^{-1}}$, where $\alpha(\boldsymbol{\mu}) > 0$ is the parameter-dependent coercivity constant (8). We estimate

$$\sup_{v \in X(\Omega)} \frac{\mathcal{G}^k(v)}{\|v\|_{H^1}} \leqslant \varepsilon \left\|(\nu_{\text{coarse}} - \nu_{\text{fine}})\nabla\big(u^k - u^{k-1}\big)\right\|_{L^2} \leqslant 2\varepsilon d \delta^{\text{tol}}_{\text{coarse}} \cdot \left\|u^k - u^{k-1}\right\|_{H^1}, \tag{9}$$

where in the last step we have used the fact that

$$\left\|\nu_{\text{coarse}}(\cdot,\boldsymbol{\mu}) - \nu_{\text{fine}}(\cdot,\boldsymbol{\mu})\right\|_{L^\infty} \leqslant \delta^{\text{tol}}_{\text{coarse}} + \delta^{\text{tol}}_{\text{fine}} \leqslant 2\delta^{\text{tol}}_{\text{coarse}}, \quad \text{for all } \boldsymbol{\mu} \in \mathcal{D} \tag{10}$$

according to assumptions (4), (5), and (6),

$$\left\|u^{k+1}-u^k\right\|_{H^1} \leqslant \frac{2\varepsilon d\delta^{\mathrm{tol}}_{\mathrm{coarse}}}{\alpha(\boldsymbol{\mu})}\left\|u^k-u^{k-1}\right\|_{H^1} \leqslant \frac{2\varepsilon d\delta^{\mathrm{tol}}_{\mathrm{coarse}}}{\alpha_{\mathrm{LB}}}\left\|u^k-u^{k-1}\right\|_{H^1}. \tag{11}$$

Thus the sequence converges in $X(\Omega)$ to the solution of (1) when $\delta^{\mathrm{tol}}_{\mathrm{coarse}} < \frac{\alpha_{\mathrm{LB}}}{2\varepsilon d}$ by contractivity. □

3. Numerical implementation

For computational purposes the exact solutions of affine problems (P0) and (Pk) are approximated with the finite element method. Given a subspace $X_h(\Omega) \subset X(\Omega)$ of piecewise polynomial continuous functions defined on a discrete, regular mesh, we seek a sequence of discrete approximations $u_h^k \in X_h(\Omega)$, $k = 1, 2, \ldots$, s.t.

$$\mathcal{A}_{\mathrm{coarse}}\big(u_h^k, v_h; \boldsymbol{\mu}\big) + \mathcal{B}\big(u_h^k, v_h; \boldsymbol{\mu}\big) = \mathcal{F}(v_h; \boldsymbol{\mu}) + \mathcal{A}_{\mathrm{coarse}}\big(u_h^{k-1}, v; \boldsymbol{\mu}\big) - \mathcal{A}_{\mathrm{fine}}\big(u_h^{k-1}, v; \boldsymbol{\mu}\big) \tag{12}$$

for all $v_h \in X_h(\Omega)$. We assume the dimension of the FE space, $\dim(X_h) = \mathcal{N}$, is chosen large enough so that the FE solution is an accurate representation of the true solution. The reduced basis method detailed in Appendix A can then be applied to obtain a reduced linear system of size $N \times N$

$$\left(\sum_{q=1}^{Q_{\mathrm{coarse}}} \Theta^q_{\mathrm{coarse}}(\boldsymbol{\mu})\underline{A}^q_N + \sum_{q=1}^{Q_b} \Theta^q_b(\boldsymbol{\mu})\underline{B}^q_N\right)u^k_N = \left(\sum_{q=1}^{Q_f} \Theta^q_f(\boldsymbol{\mu})F^q\right) + \left(\sum_{q=1}^{Q_{\mathrm{coarse}}} \Theta^q_{\mathrm{coarse}}(\boldsymbol{\mu})\underline{A}^q_N - \sum_{q=1}^{Q_{\mathrm{fine}}} \Theta^q_{\mathrm{fine}}(\boldsymbol{\mu})\underline{A}^q_N\right)u^{k-1}_N, \tag{13}$$

i.e. the reduced basis version of Eq. (Pk). We have assumed here that the coarse- and fine-level approximations are hierarchical, that is, $\nu^q_f(\boldsymbol{x}) \equiv \nu^q_c(\boldsymbol{x})$ for all $q \leqslant Q_c$. The reduced basis system matrices $\underline{A}^q_N$ (Q_{fine} in total), $\underline{B}^q_N$ (Q_b in total), and the vectors F^q (Q_f in total) need to be assembled and stored in the offline stage. This is the only part of the offline stage that has complexity depending on Q_{fine}. The rest of the offline stage depends only on Q_{coarse}. In the online stage, starting from an initial guess such as $u^0_N = [1, 0, 0, \ldots]^t$ we evaluate the coefficient functions depending only on the parameter, use the stored system matrices and right-hand sides to assemble the system (13), and solve to obtain the next iterate u^1_N. This fixed-point iteration is continued until a sufficient tolerance is achieved, $|u^K_N - u^{K-1}_N| < \mathrm{TOL}$. We can then reconstruct an approximation to the finite element solution of (7) as $u_h \approx \sum_{n=1}^N [u^K_N]_n \Phi_n(\boldsymbol{x})$, where Φ_n are the orthonormalized reduced basis functions.

4. Test case: airfoil in thermal flow

Our test case involves the design of an airfoil in an exterior thermal flow that was considered in [7]:

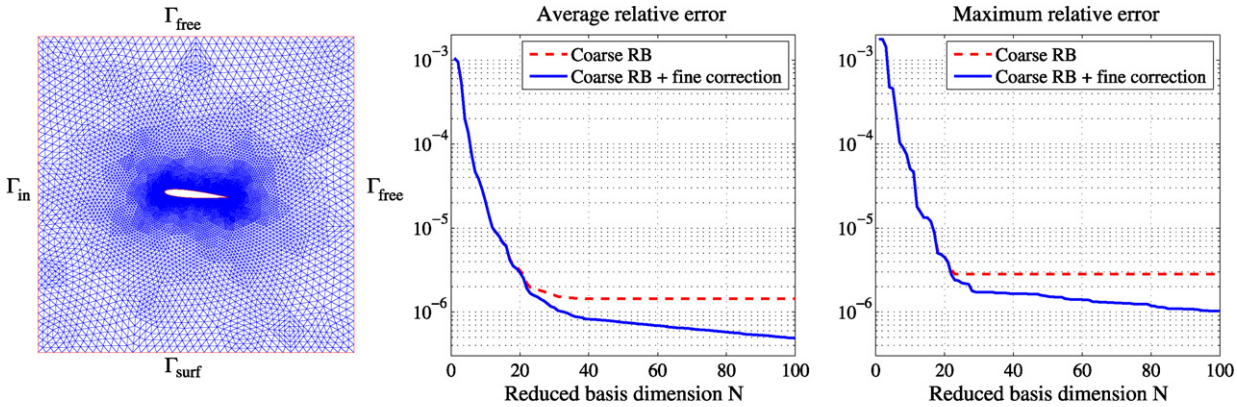
$$\int_{\Omega_o(\boldsymbol{\mu})} \big(\varepsilon\nabla u \cdot \nabla v + v\boldsymbol{w}(\boldsymbol{x}) \cdot \nabla u\big)\,\mathrm{d}\Omega_o = \int_{\Omega_o(\boldsymbol{\mu})} f v\,\mathrm{d}\Omega_o \quad \forall v \in H^1\big(\Omega_o(\boldsymbol{\mu})\big)$$
$$\text{with } u = T_0 \quad \text{on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{free}}, \qquad u = T_1 \quad \text{on } \Gamma_{\mathrm{surf}}, \qquad u = T_2 \quad \text{on the airfoil.} \tag{14}$$

The reference geometry Ω is shown in Fig. 1(a). We parametrized the geometry around the airfoil $\Omega_O(\boldsymbol{\mu}) = T(\Omega, \boldsymbol{\mu})$ by free-form deformations (FFD) [8]: a 6×6 lattice of control points was placed around the airfoil and the closest four control points are allowed to move in the x_2-direction. This results in a polynomial geometric map with $P = 4$ parameters $\boldsymbol{\mu} = \{\mu_p, p = 1, \ldots, P\}$, representing the vertical displacements by the parameters range $\mathcal{D} = [-0.2, 0.2]^4$. For the FEM computations $\mathcal{N} = 15{,}718$ degrees of freedom were used.

The advective coefficient matrix is affinely parameterized as per assumptions. We chose $\varepsilon^{-1} = 100 = \mathrm{Pe}$ (convection dominated flow). A uniform lower bound for the coercivity constant $\alpha(\boldsymbol{\mu}) > 0.1$ then holds over the parameter range and the criterion for the coarse-level tolerance is $\delta^{\mathrm{tol}}_{\mathrm{coarse}} < 2.5\mathrm{e}{-4}$. This tolerance was achieved by EIM in the example when $Q_c = 76$. For the fine-level expansion we used $\delta^{\mathrm{tol}}_{\mathrm{fine}} = 1\mathrm{e}{-7}$, which gives $Q_f = 644$ terms. The advective part was affinely decomposed with $Q_b = 11$ terms.

The cost of the offline stage is dominated by the construction of the lower bounds for the coercivity constant $\alpha(\boldsymbol{\mu})$. This involves the solution of $2(Q_b + Q_c)$ initial eigenproblems of size $\mathcal{N} \times \mathcal{N}$, and $K_{\max}$ steps of the algorithm each involves a linear programming problems of size $O(Q_b + Q_c)$. In addition we have an operation count of $O(\mathcal{N}K_{\max}(Q_b + Q_c))$, see [3]. Thus at the coarse level the SCM has to perform 87% fewer initial eigensolves, and the cost of each iteration is similarly reduced. For this problem the SCM terminated after $K_{\max} = 3$ steps. The coarse-level error bounds together with a training set of size $|\Xi_{\mathrm{train}}| = 1000$ were used to select a reduced basis with a total of $N_{\max} = 100$ basis functions. The certificate obtained for the coarse-level problem was $\Delta_N(\boldsymbol{\mu}) \leqslant 1\mathrm{e}{-5}$ for all $\boldsymbol{\mu} \in \mathcal{D}$.

In the online stage the reduced problem was solved with the coarse-level RB as an initial guess. Then the preassembled fine-level structures were used to drive the fixed-point correction iteration. Typically 4–5 fixed-point iterations were needed for convergence to $\mathrm{TOL} = 1\mathrm{e}{-9}$. The error of the reduced basis approximation when compared to the full FEM solution is displayed in Fig. 1(b). We can observe a *plateau effect* [2], i.e. the convergence of the approximation stagnates early due to



(a) Computational geometry (b) Error $||u_h - u_h^N||/||u_h||$ of RB approximation over a random sample of $\boldsymbol{\mu} \in \mathcal{D}$

Fig. 1. Finite element mesh on the computational reference domain and the reduced basis approximation errors.

replacing the non-affine diffusive tensor with its coarse approximation. The fine-level correction reduced the plateau effect while maintaining the certified error bounds at the coarse level.

5. Conclusions

The cost of certifying the solutions of the reduced basis method for non-affinely parameterized problems can be considerably reduced by adopting a two-level strategy. We demonstrated the proposed method on a geometrically parameterized advection–diffusion equation. The savings in the offline stage were 87%, mainly in terms of the number of eigensolutions needed for the successive constraint method. The fine-level correction improved the error of the final reduced basis solution and reduced the plateau effect typically observed in non-affinely parameterized problems when the empirical interpolation tolerance is not sufficiently small. The certified error bounds were recovered from the coarse-level problem.

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Appendix A. Reduced basis method for parametric PDEs

The standard Galerkin finite element (FE) approximation of (1) is to find $u_h \in X_h(\Omega)$ s.t. $\mathcal{A}(u_h, v; \boldsymbol{\mu}) + \mathcal{B}(u_h, v; \boldsymbol{\mu}) = \mathcal{F}(v; \boldsymbol{\mu})$ for all $v \in X_h(\Omega)$, where X_h is an FE space constructed by using e.g. piecewise linear shape functions on a discrete mesh. We assume $\dim(X_h(\Omega)) = \mathcal{N}$ is large enough so that the repeated assembly and solution of the FE system is prohibitively expensive for a multi-query context. In order to find an approximation to u_h in an efficient and reliable way, we use Galerkin projection on a reduced subspace of basis functions. Let $\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^N$ be a collection of parameter vectors and define the reduced basis approximation space as $X_h^N := \mathrm{span}\{u_h(\boldsymbol{\mu}^n)\colon\ n = 1, \ldots, N\}$, where each $u_h(\boldsymbol{\mu}^n) \in X_h$ is an FE solution for a given parameter value $\boldsymbol{\mu}^n$. The reduced basis formulation reads as follows: find $u_h^N \in X_h^N$ s.t. $\mathcal{A}(u_h^N, v; \boldsymbol{\mu}) + \mathcal{B}(u_h^N, v; \boldsymbol{\mu}) = \mathcal{F}(v)$, for all $v \in X_h^N$. If the parametric bilinear forms are affinely parametrized [6], that is to say of the form $\mathcal{A}(u, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\mathcal{A}^q(u, v)$ and $\mathcal{B}(u, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\mathcal{B}^q(u, v)$, the solution of the reduced basis problem splits into two stages. In the so-called *offline stage* we assemble and store once and for all the parameter-independent system matrices $\underline{A}^q$ and $\underline{B}^q$ of components $[\underline{A}^q]_{m,n} = \mathcal{A}^q(\Phi_n, \Phi_m)$ and $[\underline{B}^q]_{m,n} = \mathcal{B}^q(\Phi_m, \Phi_n)$ using the global reduced basis functions Φ_k, and similarly for the right-hand sides. Then in the *online stage* for a given parameter $\boldsymbol{\mu}$ the parametric coefficients $\Theta_a^q(\boldsymbol{\mu})$, $\Theta_b^q(\boldsymbol{\mu})$ are evaluated and the reduced basis matrices $\underline{A}_N = \sum_{q=1}^{Q_a} \Theta_a^q(\boldsymbol{\mu})\underline{A}^q$, $\underline{B}_N = \sum_{q=1}^{Q_b} \Theta_b^q(\boldsymbol{\mu})\underline{B}^q$ are assembled, and similarly for the right-hand side. This linear system of dimension $N \times N$ is dense, but inexpensive to solve: the online complexity is independent of the FE solution dimension $\mathcal{N}$ [5].

To have confidence in our reduced model we require an efficient and computable a posteriori error estimator $\Delta_N(\boldsymbol{\mu})$ that measures the error: $\|u_h(\boldsymbol{\mu}) - u_h^n(\boldsymbol{\mu})\| \leqslant \Delta_n(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in \mathcal{D}$. In this case we say that the reduced model is *certified*. The standard estimator is based on the residual of the reduced basis solution $\Delta_N(\boldsymbol{\mu}) := \|\mathcal{F}(\cdot; \boldsymbol{\mu}) - \mathcal{A}(u_h^N, \cdot; \boldsymbol{\mu}) - \mathcal{B}(u_h^N, \cdot; \boldsymbol{\mu})\|_{X_h'(\Omega)}/\alpha_{\mathrm{LB}}(\boldsymbol{\mu})$, where $\alpha_{\mathrm{LB}}(\boldsymbol{\mu})$ is an online computable lower bound estimate for the parametric coercivity constant of the problem. To construct $\alpha_{\mathrm{LB}}(\boldsymbol{\mu})$ the successive constraint method (SCM) [3] can be used. For the details of implementing $\Delta_N(\boldsymbol{\mu})$ we refer to [6,4].

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