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Group Theory/Lie Algebras

Exterior powers of the reflection representation in the cohomology of Springer fibres

Les puissances extérieures de la représentation géométrique dans la cohomologie des fibres de Springer

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ABSTRACT

Let $H^*(\mathcal{B}_e)$ be the cohomology of the Springer fibre for the nilpotent element e in a simple Lie algebra $\mathfrak{g}$. Let $\Lambda^i V$ denote the ith exterior power of the reflection representation of W. We determine the degrees in which $\Lambda^i V$ occurs in the graded representation $H^*(\mathcal{B}_e)$, under the assumption that e is regular in a Levi subalgebra and satisfies a certain extra condition which holds automatically if $\mathfrak{g}$ is of type A, B, or C. This partially verifies a conjecture of Lehrer and Shoji.

RÉSUMÉ

Soit $H^*(\mathcal{B}_e)$ la cohomologie de la fibre de Springer pour l'élément nilpotent e de l'algèbre de Lie simple $\mathfrak{g}$. Soit $\Lambda^i V$ la i-ème puissance extérieure de la représentation géométrique de W. Nous trouvons les degrés des contributions de $\Lambda^i V$ à la représentation graduée $H^*(\mathcal{B}_e)$, si e est régulier dans une sous-algèbre de Levi et satisfait à une autre condition qui est vraie si $\mathfrak{g}$ est de type A, B, ou C. Ce résultat démontre partiellement une conjecture de Lehrer et Shoji.

1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank ℓ. Let W denote the Weyl group of $\mathfrak{g}$, and let V be the reflection representation of W. It is well known that the exterior powers $\Lambda^i V$, for $i = 0, 1, \ldots, \ell$, are inequivalent irreducible representations of W, each of which is self-dual.

Let e be a nilpotent element of $\mathfrak{g}$. The Springer fibre $\mathcal{B}_e$ is the variety of Borel subalgebras of $\mathfrak{g}$ containing e. Let $H^*(\mathcal{B}_e)$ denote the graded cohomology ring of $\mathcal{B}_e$ with complex coefficients; the cohomology lives solely in even degrees, so $H^*(\mathcal{B}_e)$ is commutative. We have the Springer representation of W on each $H^{2j}(\mathcal{B}_e)$ (see [5], [2, Chapter 9]). Let s (depending on e) denote the multiplicity of the irreducible representation V in the total representation $H^*(\mathcal{B}_e)$. Let $m_1, m_2, \ldots, m_s$ be the multiset of nonnegative integers, listed in increasing order, which are the halved degrees of the occurrences of V in the graded representation $H^*(\mathcal{B}_e)$. That is, we have by definition $\sum_j \dim(H^{2j}(\mathcal{B}_e) \otimes V)^W q^j = q^{m_1} + q^{m_2} + \cdots + q^{m_s}$.

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In the special case $e = 0$, it is well known (see [5, §5]) that $H^*(\mathcal{B})$ is isomorphic to the coinvariant algebra $C^*(W)$ of W, that $s = \ell$, and that $m_1, \ldots, m_\ell$ are the exponents of W. More generally, if e is a regular nilpotent in a Levi subalgebra of semisimple rank r, then it was proved by Lehrer and Shoji in [3, Theorem 2.4] (see also [9]) that $s = \ell - r$ and that $m_1, \ldots, m_s$ are the coexponents of the corresponding parabolic hyperplane arrangement, in the sense of Orlik and Solomon. See [8] for some other related interpretations of these coexponents. For $\mathfrak{g}$ of classical type and general e, the numbers $m_1, m_2, \ldots, m_s$ were calculated by Spaltenstein in [9, Propositions 1.6–1.9].

Lehrer and Shoji conjectured that, at least in the parabolic case which they considered, the occurrences of each exterior power $\Lambda^i V$ in $H^*(\mathcal{B}_e)$ were also controlled in a natural way by $m_1, m_2, \ldots, m_s$.

Conjecture 1.1. *(See [3, Conjecture 8.3].) Suppose that e is a regular nilpotent in a Levi subalgebra. Then for any $i = 0, 1, \ldots, \ell$, we have $\sum_j \dim(H^{2j}(\mathcal{B}_e) \otimes \Lambda^i V)^W q^j = e_i(q^{m_1}, q^{m_2}, \ldots, q^{m_s})$ (the ith elementary symmetric polynomial in $q^{m_1}, q^{m_2}, \ldots, q^{m_s}$, which is defined to be zero if $i > s$).*

The $e = 0$ case of this conjecture had already been proved by Solomon in [6]; indeed, he proved the stronger statement that the algebra $(C^*(W) \otimes \Lambda^* V)^W$ is a free exterior algebra on $(C^*(W) \otimes V)^W$.

The main result of this Note is the following generalization of Solomon's result, which implies various cases of Conjecture 1.1:

Theorem 1.2. *Suppose that e is regular in a Levi subalgebra of $\mathfrak{g}$, and define s and $m_1, \ldots, m_s$ as above. Also suppose that there is a parabolic subgroup W_K of W such that the following two conditions hold:*

(1) *There exist invariant polynomials $f_1, f_2, \ldots, f_s \in (S^* V)^W$, homogeneous of degrees $m_1 + 1, m_2 + 1, \ldots, m_s + 1$, whose restrictions to the reflection representation V_K of W_K form a set of fundamental invariants for W_K.*
(2) *The nilpotent orbit of e intersects the nilradical of the parabolic subalgebra $\mathfrak{p}_K$ associated to W_K.*

(See Section 2 for the definitions of V_K and $\mathfrak{p}_K$.) Then the algebra $(H^*(\mathcal{B}_e) \otimes \Lambda^* V)^W$ is a free exterior algebra on $(H^*(\mathcal{B}_e) \otimes V)^W$. More precisely, the natural homomorphism $\psi : \Lambda^*((H^*(\mathcal{B}_e) \otimes V)^W) \to (H^*(\mathcal{B}_e) \otimes \Lambda^* V)^W$ is an isomorphism.

Here the domain and codomain of ψ are $(\mathbb{N} \times \mathbb{N})$-graded algebras over $\mathbb{C}$, where the (i, j)-components are $\Lambda^i((H^{2j}(\mathcal{B}_e) \otimes V)^W)$ and $(H^{2j}(\mathcal{B}_e) \otimes \Lambda^i V)^W$ respectively, and in both cases the algebra multiplication is graded-commutative with respect to the $\mathbb{N}$-grading labelled by i; the homomorphism ψ is induced by the inclusion of the subspace $(H^*(\mathcal{B}_e) \otimes V)^W$ in $(H^*(\mathcal{B}_e) \otimes \Lambda^* V)^W$. Since the graded degrees of this subspace are $(1, m_1), (1, m_2), \ldots, (1, m_s)$, the statement that ψ is an isomorphism implies Conjecture 1.1.

Simple calculations[2] verify the following results:

Proposition 1.3. *If $\mathfrak{g}$ is of type A, then the assumptions of Theorem 1.2 hold for any e.*

Proposition 1.4. *If $\mathfrak{g}$ is of type B or C, then the assumptions of Theorem 1.2 hold for any e which is regular in a Levi subalgebra.*

Hence Conjecture 1.1 is proved in types A–C. By contrast, suppose that $\mathfrak{g}$ is of type D_4 and e has Jordan type $(3^2 1^2)$ in the natural representation on $\mathbb{C}^8$. Then e is regular in a Levi subalgebra of type A_2, but we have $m_2 = 2$ and there are no W-invariant polynomials of degree 3, so condition (1) of Theorem 1.2 cannot be satisfied.

2. Proof of Theorem 1.2

Continue the notation of the introduction. Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra and Borel subalgebra of $\mathfrak{g}$, and let $\Pi \subset \Phi^+ \subset \Phi$ be the corresponding set of simple roots, positive roots, and roots. We identify W with the subgroup of $GL(\mathfrak{h})$ generated by the simple reflections s_α for $\alpha \in \Pi$; the reflection representation V of W is merely $\mathfrak{h}$ itself.

Let $J \subseteq \Pi$ be a subset of size r, and set $s = \ell - r$. We have a Levi subalgebra $\mathfrak{l}_J$ and parabolic subalgebra $\mathfrak{p}_J$ containing $\mathfrak{h}$ and $\mathfrak{b}$ respectively, a parabolic subsystem Φ_J of Φ, and a parabolic subgroup W_J of W. Define $V^J = \bigcap_{\alpha \in J} \ker(\alpha) = V^{W_J}$. We write V_J for the unique W_J-invariant complement to V^J in V, which is the reflection representation of W_J. Note that $\dim V^J = s$ and $\dim V_J = r$. Let $\mathcal{A}^J$ and $\mathcal{A}_J$ be the hyperplane arrangements in V^J and V_J respectively induced by the root hyperplanes in V.

We assume for the remainder of the section that e is parabolic of type J, meaning that the orbit of e contains the regular nilpotent elements of $\mathfrak{l}_J$. As mentioned in the introduction, Lehrer and Shoji proved in this case that $\sum_j \dim(H^{2j}(\mathcal{B}_e) \otimes V)^W q^j = q^{m_1} + q^{m_2} + \cdots + q^{m_s}$, where $m_1, \ldots, m_s$ are the coexponents of the arrangement $\mathcal{A}^J$. (See [3, Theorem 2.4]; the missing case in type D is covered by the results of Spaltenstein [9].)

An important special feature of the parabolic case is Lusztig's Induction Theorem for Springer representations.

[2] The details may be found in the preprint version of this Note, arXiv:1001.3164.

Theorem 2.1. *(See [4].) The representation of W on $H^*(\mathcal{B}_e)$, neglecting the grading, is isomorphic to the induction $\mathrm{Ind}_{W_J}^W(\mathbb{C})$ of the trivial representation of W_J.*

Corollary 2.2. *We have $\dim(H^*(\mathcal{B}_e) \otimes \Lambda^*V)^W = 2^s$, so the domain and codomain of ψ have the same dimension.*

Proof. By Frobenius Reciprocity, we know that $\dim(\mathrm{Ind}_{W_J}^W(\mathbb{C}) \otimes \Lambda^*V)^W = \dim(\Lambda^*V)^{W_J}$. From the fact that $(\Lambda^*V)^{s_\alpha} = \Lambda^*(\ker(\alpha))$ for all $\alpha \in J$ one deduces $(\Lambda^*V)^{W_J} = \Lambda^*(V^J)$, and the corollary follows. □

So to prove Theorem 1.2, it suffices to show that the homomorphism ψ is injective. Since the domain is an exterior algebra, this will follow if we can show that $\psi(\Lambda^s((H^*(\mathcal{B}_e) \otimes V)^W)) \neq 0$.

Using the W-equivariant isomorphism of V with its dual, we will identify the symmetric algebra S^*V with the ring of polynomial functions on V. It is well known that the invariant subring $(S^*V)^W$ is freely generated by ℓ homogeneous polynomials, called fundamental invariants for W. The coinvariant algebra $C^*(W)$ of W is the quotient S^*V/I, where I is the ideal generated by these fundamental invariants.

Now there is a canonical (degree-doubling) W-equivariant algebra homomorphism $S^*V \to H^*(\mathcal{B})$ which identifies $H^*(\mathcal{B})$ with $C^*(W)$ (see [5, §5]). Composing this with the natural homomorphism $H^*(\mathcal{B}) \to H^*(\mathcal{B}_e)$, which is W-equivariant by [9, Lemma 1.4] (this fact in characteristic p was [1, Theorem 1.1]), we obtain a W-equivariant homomorphism $\varphi : S^*V \to H^*(\mathcal{B}_e)$. Note that the image of φ is contained in the subspace $H^*(\mathcal{B}_e)^{A(e)}$ of invariants for the component group of the centralizer of e in the adjoint group of $\mathfrak{g}$; in particular, φ is not surjective in general. However, it may happen that the induced map $(S^*V \otimes V)^W \to (H^*(\mathcal{B}_e) \otimes V)^W$ is surjective even if φ is not. We will see that this occurs under the assumptions of Theorem 1.2, which means that a calculation with polynomials on V suffices to prove what we want.

Henceforth we let $K \subseteq \Pi$ be a subset satisfying conditions (1) and (2) of Theorem 1.2. Note that condition (1) entails $|K| = s$. Choose a basis $v_1, v_2, \ldots, v_\ell$ of V such that $v_1, \ldots, v_s$ is a basis of V_K. Since the exterior derivative $S^*V \to S^*V \otimes V : f \mapsto \sum \frac{\partial f}{\partial v_j} \otimes v_j$ is W-equivariant, we have the following s elements of $(H^*(\mathcal{B}_e) \otimes V)^W$:

$$\sum_{j=1}^{n} \varphi\left(\frac{\partial f_1}{\partial v_j}\right) \otimes v_j, \sum_{j=1}^{n} \varphi\left(\frac{\partial f_2}{\partial v_j}\right) \otimes v_j, \ldots, \sum_{j=1}^{n} \varphi\left(\frac{\partial f_s}{\partial v_j}\right) \otimes v_j.$$

We can prove both that these form a basis of $(H^*(\mathcal{B}_e) \otimes V)^W$, and that $\psi(\Lambda^s((H^*(\mathcal{B}_e) \otimes V)^W)) \neq 0$ as required, by proving the single fact that $\varphi(\Delta) \neq 0$, where Δ is the determinant of the $s \times s$ matrix $(\frac{\partial f_a}{\partial v_b})$, with a and b ranging from 1 to s. Since $v_1, \ldots, v_s$ span the reflection representation of W_K, we have $w\Delta = \varepsilon(w)\Delta$ for all $w \in W_K$. This forces Δ to be divisible by the polynomial $\pi_K = \prod_{\beta \in \Phi_K^+} \beta$. Condition (1) tells us that on restricting to V_K, Δ becomes the Jacobian matrix of the fundamental invariants of W_K, which is well known to be a nonzero scalar multiple of the restriction to V_K of π_K (see [10]). So Δ is a nonzero scalar multiple of π_K, and it suffices to prove that $\varphi(\pi_K) \neq 0$.

By condition (2), we may suppose that e lies in the nilradical of $\mathfrak{p}_K$. Then any Borel subalgebra contained in $\mathfrak{p}_K$ must contain e, so we have an inclusion $\mathcal{B}^K \hookrightarrow \mathcal{B}_e$, where $\mathcal{B}^K$ denotes the variety of Borel subalgebras contained in $\mathfrak{p}_K$, which can be identified with the flag variety of $\mathfrak{l}_K$. Hence it suffices to prove that π_K is not in the kernel of the composition $S^*V \to H^*(\mathcal{B}) \to H^*(\mathcal{B}^K)$. But this composition is the canonical homomorphism identifying $C^*(W_K)$ with $H^*(\mathcal{B}^K)$, which maps π_K to a generator of the top-degree cohomology of $\mathcal{B}^K$ (compare [1, Proposition 1.4], which uses exactly this argument in the case when e lies in the Richardson orbit of $\mathfrak{p}_K$). This completes the proof of Theorem 1.2.

Acknowledgements

The main result of this Note, Theorem 1.2, dates from 1997, when I was a student at the University of Sydney, supervised by Gus Lehrer. As the reader will observe, it is indebted to Lehrer's ideas, and I thank him for his help and encouragement. I did not publish this result at the time, since it did not prove the motivating Conjecture 1.1 in general. Recently Eric Sommers [7] has completed the proof of Conjecture 1.1 by a different method, and also removed the assumption that e is regular in a Levi subalgebra. I thank him for his interest in my old result, and for the suggestion that it be published to supply part of the general proof.

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