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Differential Geometry

Liouville type theorem and uniform bound for the Lichnerowicz equation and the Ginzburg–Landau equation ☆

Un théorème de type de Liouville et borne inférieure des solutions régulières pour l'équation de Lichnerowicz et pour l'équation de Ginsburg–Landau

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ABSTRACT

In this Note, we prove the Liouville type result for smooth positive solutions to the Lichnerowicz equation in R^n. Using the same method, we also give the uniform bound of the smooth solutions to Ginzburg–Landau equation in the whole space. Similar results on a complete non-compact Riemannian manifold with the Ricci curvature bounded from below are also considered.

RÉSUMÉ

Dans cette Note on démontre un résultat de type de Liouville des solutions régulières pour l'équation de Lichnerowicz dans R^n. En utilisant la même méthode on détermine également une borne uniforme inférieure des solutions régulières pour l'équation de Ginzburg–Landau dans tout l'espace. Des résultats analogues sont donnés dans le cas d'une variété riemannienne non compacte complète de courbure de Ricci bornée inférieurement.
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1. Introduction

In the recent interesting paper [2], the authors have proved an existence result for the Lichnerowicz equation on closed Riemannian manifolds by the mini-max method. Then Druet and Hebey [1] have further considered the stability problems for the Lichnerowicz equation on closed Riemannian manifolds. It is also interesting to consider the Lichnerowicz equation in complete non-compact Riemannian manifolds. In our previous paper [3], we have proposed the question if the Liouville type result is true for smooth positive solutions to the Lichnerowicz equation in R^n. Using the idea from Redheffer (see the paper of Serrin [4]), we confirm this assertion.

Our Liouville type result follows:

Theorem 1. *Let $u > 0$ be a smooth positive solution to the Lichnerowicz equation in R^n:*

$$\Delta u = u^{p-1} - u^{-p-1}, \quad in\ R^n, \tag{1}$$

where $p > 1$. Then $u = 1$.

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The proof of this result is simple and we give it here: Recall that we have already showed that every smooth positive solution to (2) is bounded [3]. Let $f(u) = u^q - u^{-p-1}$ for some $q > 1$ and $p > 1$. Then $f(u)$ is monotone non-decreasing.

For any fixed $\epsilon > 0$ and arbitrary point $x_0 \in R^n$, we let

$$w(x) := w_R(x) = u(x) - u(x_0) + \epsilon - \epsilon|x - x_0|^2.$$

Since $w(x) \to -\infty$ as $|x| \to \infty$ and $w(x_0) = \epsilon$, we know that there is a point $y \in R^n$ such that

$$w(y) = \max_{R^n} w(x) \geqslant \epsilon.$$

By this, we have $u(y) \geqslant u(x_0) - \epsilon$. Since $0 \geqslant \Delta w(y) = \Delta u(y) - 2n\epsilon$, we get

$$2n\epsilon \geqslant \Delta u(y) \geqslant f\big(u(y)\big).$$

Since the derivative $f'(u) > 0$ for $u > 0$, we have

$$f\big(u(y)\big) \geqslant f\big(u(x_0) - \epsilon\big).$$

Then we have

$$2n\epsilon \geqslant f\big(u(x_0) - \epsilon\big).$$

Sending $\epsilon \to 0$ we have

$$f\big(u(x_0)\big) \leqslant 0.$$

Similarly, the minimum of the function

$$v_R(x) = u(x) - u(x_0) - \epsilon + \epsilon|x - x_0|^2,$$

we can show that

$$f\big(u(x_0)\big) \geqslant 0.$$

Hence we have $f(u(x_0)) = 0$, which implies that $u(x_0) = 1$. Since $x_0 \in R^n$ is arbitrary, we know that $u = 1$. This completes the proof of Theorem 1.

We remark that the similar Liouville type result is also true for smooth positive solutions for the Lichnerowicz equation in a complete non-compact Riemannian manifold with the Ricci curvature bounded from below.

Theorem 2. *Let $u > 0$ be a smooth positive solution to the Lichnerowicz equation in the complete non-compact Riemannian manifold (M^n, g) with the Ricci curvature bounded from below*:

$$\Delta u = u^{p-1} - u^{-p-1}, \quad \text{in } M^n, \tag{2}$$

where $p > 1$. Then $u = 1$.

The only difference is that we use the existence of a smooth function $\phi \in C^2(M)$ [5] such that

$$c^{-1}\big(1 + d(x, x_0)\big) \leqslant \phi(x) \leqslant C\big(1 + d(x, x_0)\big), \qquad \big|\nabla\phi(x)\big| \leqslant C,$$

and $\Delta\phi(x) \leqslant C$, where $C > 0$ is a uniform constant, $d(x, x_0)$ is the distance function of (M, g) between two points x and x_0. We just replace $\epsilon|x - x_0|^2$ by $\epsilon^2\phi(x)$.

In the remaining part of this paper, we show that this kind idea can be used to prove the bounded-ness of the smooth solutions to the famous Ginzburg–Landau equation in R^n. Precisely, we consider the smooth solutions u to the Ginzburg–Landau equation

$$\Delta u + u\big(1 - |u|^2\big) = 0, \quad \text{in } R^n \tag{3}$$

where $u : R^n \to R^N$. We shall prove the following result due to H. Brezis (and I thank Dr. Yuxin Ge for telling me this result in his note in Paris in 2004):

Theorem 3. *Assume that $u \in C^2(R^n)$ is a smooth solution to* (3)*. Then we have $|u(x)| \leqslant 1$ in R^n.*

Similarly, we have

Theorem 4. *Consider the Ginzburg–Landau equation on the complete non-compact Riemannian manifold* (M, g) *with the Ricci curvature bounded from below*

$$\Delta u + u\left(1 - |u|^2\right) = 0, \quad \text{in } M^n \tag{4}$$

where $u : M^n \to R^N$.

Assume that $u \in C^2(M^n)$ *is a smooth solution to* (4). *Then we have* $|u(x)| \leqslant 1$ *in* M^n.

Since the argument of Theorem 4 is similar to Theorem 3, we omit the proof.

2. Proof of Theorem 3

We firstly show that u is bounded in R^n.

For any unit vector $\nu \in S^{N-1}$, we define $v = \nu \cdot u$. Then we have

$$\Delta v + v\left(1 - |u|^2\right) = 0, \quad \text{in } R^n.$$

Let $V = v^2$. Then, using $|u|^2 \geqslant V$, we have

$$\Delta V \geqslant 2v\Delta v \geqslant -2V(1 - V), \quad \text{in } R^n.$$

Given any small $R > 0$ and large $\alpha > 1$. Let

$$w(x) := w_R(x) = \left(R^2 - |x - x_0|^2\right)^{-\alpha}.$$

By direct computation, we can see that

$$\Delta w + 2w(1 - w) \leqslant 0, \quad \text{in } B_R(x_0).$$

Since $w(x) = +\infty$, we get by the comparison lemma that

$$v^2(x) = V(x) \leqslant w(x), \quad \text{in } B_R(x_0).$$

Then we have some uniform constant $C(R) > 0$ such that

$$\left|v(x)\right| \leqslant C(R), \quad \text{in } B_{R/2}(x_0).$$

Since x_0 and $\nu \in S^{N-1}$ are arbitrary, we have that

$$\left|u(x)\right| \leqslant C(R), \quad \text{in } R^n.$$

We now prove $|u(x)| \leqslant 1$ on R^n. We argue by contradiction. Let $F(v) = v(1 - v^2)$.

Case 1. Assume that we have a point $x_0 \in R^n$ such that $v(x_0) > 1$. Note that for $v(x) > 0$, we have

$$\Delta v + v\left(1 - v^2\right) \geqslant 0, \quad \text{in } R^n.$$

For small $\epsilon > 0$, we let

$$W_1(x) := v(x) - v(x_0) + \epsilon - \epsilon|x - x_0|^2.$$

Since $W_1(x) \to -\infty$ as $|x| \to \infty$ and $W_1(x_0) = \epsilon$, we know that there is a point $y \in R^n$ such that

$$W_1(y) = \max_{R^n} W_1(x) \geqslant \epsilon.$$

By this, we have $v(y) \geqslant v(x_0) - \epsilon > 0$.

Since $0 \geqslant \Delta W_1(y) = \Delta v(y) - 2n\epsilon$, we get

$$2n\epsilon \geqslant \Delta v(y) \geqslant -F\left(v(y)\right).$$

Since the derivative $-F'(v) > 0$ for $|v| > 1$, we have

$$-F\left(v(y)\right) \geqslant -F\left(v(x_0) - \epsilon\right).$$

Then we have

$$2n\epsilon \geqslant -F\left(v(x_0) - \epsilon\right).$$

Sending $\epsilon \to 0$ we have

$$\left|v(x_0)\right|^2 v(x_0) \leqslant v(x_0)$$

which implies that $v(x_0) \leqslant 1$, a contradiction.

Case 2. Assume that we have a point $x_0 \in R^n$ such that $v(x_0) < -1$.

Note that for $v(x) < 0$, we have

$$\Delta v + v\left(1 - v^2\right) \leqslant 0, \quad \text{in } R^n.$$

For small $\epsilon > 0$, we let

$$W_2(x) := v(x) - v(x_0) - \epsilon + \epsilon|x - x_0|^2.$$

Since $W_2(x) \to +\infty$ as $|x| \to \infty$ and $W_2(x_0) = -\epsilon$, we know that there is a point $z \in R^n$ such that

$$W_2(z) = \min_{R^n} W_2(x) \leqslant -\epsilon.$$

By this, we have $v(z) \leqslant v(x_0) + \epsilon < 0$.

Since $0 \leqslant \Delta W_2(z) = \Delta v(z) + 2n\epsilon$, we get

$$-2n\epsilon \leqslant \Delta v(z) \leqslant -F\left(v(z)\right).$$

Since the derivative $-F'(v) > 0$ for $|v| > 1$, we have

$$-F\left(v(z)\right) \leqslant -F\left(v(x_0) + \epsilon\right).$$

Then we have

$$-2n\epsilon \leqslant -F\left(v(x_0) + \epsilon\right).$$

Sending $\epsilon \to 0$ we have

$$v(x_0) \leqslant \left|v(x_0)\right|^2 v(x_0),$$

which implies that $v(x_0) \geqslant -1$, again, a contradiction.

Hence $|v(x_0)| \leqslant 1$ for arbitrary $x_0 \in R^n$ and arbitrary $\nu \in S^{N-1}$. We then conclude that $|u(x)| \leqslant 1$ in R^n. This completes the proof of our Theorem 3.

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