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Probability Theory

Central limit theorem for capacities

Théorème de limite centrale pour capacités

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A B S T R A C T

In this Note, our aim is to obtain the central limit theorem for capacities induced by sublinear expectations.

R É S U M É

Le but de cette Note est d'établir un théorème central limite pour les capacités associées à une espérance sous-linéaire.

1. Introduction

Central limit theorem (CLT) is long and widely been known as a fundamental result in probability theory. It is very useful in many fields.

Recently motivated by model uncertainties in statistics, finance and economics, Peng [2–7] initiated the notion of IID random variables and the definition of G-normal distribution under sublinear expectations. He further obtained new central limit theorems (CLT) under sublinear expectations. A natural question is the following: Can the classical CLT be generalized for capacities? In this Note, adapting Peng's IID notion and applying Peng's CLT under sublinear expectations, we investigate CLT for capacities.

This Note is organized as follows: in Section 2, we give some notions and lemmas that are useful in this Note. In Section 3, we give the main result including the proof.

2. Preliminaries

We present some preliminaries in the theory of sublinear expectations. More details of this section can be found in Peng [2–7].

Definition 2.1. Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω. We assume that all constants are in $\mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of our "random variables". A nonlinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \mapsto R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geqslant Y$ then $\hat{E}[X] \geqslant \hat{E}[Y]$.
(b) Constant preserving: $\hat{E}[c] = c$.

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The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a nonlinear expectation space (compare with a probability space $(\Omega, \mathcal{F}, P)$). We are mainly concerned with sublinear expectation where the expectation $\hat{E}$ satisfies also

(c) Sub-additivity: $\hat{E}[X] - \hat{E}[Y] \leqslant \hat{E}[X - Y]$.
(d) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, $\forall \lambda \geqslant 0$.

If only (c) and (d) are satisfied, $\hat{E}$ is called a sublinear functional.

The following representation theorem for sublinear expectations is very useful (see Peng [6,7] for the proof):

Lemma 2.1. *Let $\hat{E}$ be a sublinear functional defined on $(\Omega, \mathcal{H})$, i.e.,* (c) *and* (d) *hold for $\hat{E}$. Then there exists a family $\{E_\theta\colon \theta \in \Theta\}$ of linear functionals on $(\Omega, \mathcal{H})$ such that*

$$\hat{E}[X] = \max_{\theta \in \Theta} E_\theta[X]. \tag{1}$$

If (a) *and* (b) *also hold, then E_θ are linear expectations for $\theta \in \Theta$. If we make furthermore the following assumption:* (H) *For each sequence $\{X_n\}_{n=1}^\infty \subset \mathcal{H}$ such that $X_n(\omega) \downarrow 0$ for ω, we have $\hat{E}[X_n] \downarrow 0$. Then for each $\theta \in \Theta$, there exists a unique (σ-additive) probability measure P_θ defined on $(\Omega, \sigma(\mathcal{H}))$ such that*

$$E_\theta[X] = \int_\Omega X(\omega)\, \mathrm{d}P_\theta(\omega), \quad X \in \mathcal{H}. \tag{2}$$

In this Note, we are interested in the following sublinear expectation:

$$\overline{E}[\cdot] = \sup_{Q \in \mathcal{P}} E_Q[\cdot],$$

where $\mathcal{P}$ is a set of probability measures. Let Ω be a given set and let $\mathcal{F}$ be a σ-algebra. Define $\overline{V}(A) := \overline{E}[I_A] = \sup_{Q \in \mathcal{P}} E_Q[I_A]$, $\bar{v}(A) := -\overline{E}[-I_A] = \inf_{Q \in \mathcal{P}} E_Q[I_A]$, $\forall A \in \mathcal{F}$, then $\overline{V}$ and $\bar{v}$ are two capacities.

Let $C_{l,Lip}(R^n)$ denote the space of functions φ satisfying

$$\big|\varphi(x) - \varphi(y)\big| \leqslant C\big(1 + |x|^m + |y|^m\big)|x - y| \quad \forall x, y \in R^n,$$

for some $C > 0$, $m \in N$ depending on φ and let $C_{b,Lip}(R^n)$ denote the space of bounded functions φ satisfying

$$\big|\varphi(x) - \varphi(y)\big| \leqslant C|x - y| \quad \forall x, y \in R^n,$$

for some $C > 0$ depending on φ.

The following is the notion of IID random variables under sublinear expectations introduced by Peng [2–7]:

Definition 2.2. *Independence*: Suppose that $Y_1, Y_2, \ldots, Y_n$ is a sequence of random variables such that $Y_i \in \mathcal{H}$. Random variable Y_n is said to be independent of $X := (Y_1, \ldots, Y_{n-1})$ under $\overline{E}$, if for each function $\varphi \in C_{l,Lip}(R^n)$, we have

$$\overline{E}\big[\varphi(X, Y_n)\big] = \overline{E}\big[\overline{E}\big[\varphi(x, Y_n)\big]_{x=X}\big].$$

Identical distribution: Random variables X and Y are said to be identically distributed, denoted by $X \sim Y$, if for each function $\varphi \in C_{l,Lip}(R)$, we have

$$\overline{E}\big[\varphi(X)\big] = \overline{E}\big[\varphi(Y)\big].$$

IID random variables: A sequence of random variables $\{X_n\}_{n=1}^\infty$ is said to be IID, if $X_n \sim X_1$ and X_{n+1} is independent of $Y := (X_1, \ldots, X_n)$ for each $n \geqslant 1$.

Definition 2.3 *(G-normal distribution, see Definition 10 in Peng [3]).* A random variable $\xi \in \mathcal{H}$ under sublinear expectation $\widetilde{E}$ with $\bar{\sigma}^2 = \widetilde{E}[\xi^2]$, $\underline{\sigma}^2 = -\widetilde{E}[-\xi^2]$ is called G-normal distribution, denoted by $\mathcal{N}(0; [\underline{\sigma}^2, \bar{\sigma}^2])$, if for any function $\varphi \in C_{l,Lip}(R)$, write $u(t, x) := \widetilde{E}[\varphi(x + \sqrt{t}\xi)]$, $(t, x) \in [0, \infty) \times R$, then u is the unique viscosity solution of PDE:

$$\partial_t u - G\big(\partial_{xx}^2 u\big) = 0, \quad u(0, x) = \varphi(x),$$

where $G(x) := \frac{1}{2}(\bar{\sigma}^2 x^+ - \underline{\sigma}^2 x^-)$ and $x^+ := \max\{x, 0\}$, $x^- := (-x)^+$.

The following lemma is very useful in this Note:

Lemma 2.2. *Suppose that ξ is G-normal distributed by $\mathcal{N}(0;[\underline{\sigma}^2,\bar{\sigma}^2])$. Let P be a probability measure and φ be a bounded continuous function with compact support. If $\{B_t\}_{t\geqslant 0}$ is a P-Brownian motion, then*

$$\widetilde{E}\big[\varphi(\xi)\big]=\sup_{\theta\in\Theta}E_P\Bigg[\varphi\Bigg(\int_0^1\theta_s\,\mathrm{d}B_s\Bigg)\Bigg],$$

where

$$\Theta:=\big\{\{\theta_t\}_{t\geqslant 0}\colon\ \theta_t \text{is } \mathcal{F}_t\text{-adapted process such that } \underline{\sigma}\leqslant\theta_t\leqslant\bar{\sigma}\big\},$$
$$\mathcal{F}_t:=\sigma\{B_s, 0\leqslant s\leqslant t\}\vee\mathcal{N},\quad \mathcal{N} \text{ is the collection of } P\text{-null subsets}.$$

Proof. By Proposition 54 in Denis, Hu and Peng [1], we have: for each $\varphi\in C_{b,Lip}(R)$,

$$\widetilde{E}\big[\varphi(\xi)\big]=\sup_{\theta\in\Theta}E_P\Bigg[\varphi\Bigg(\int_0^1\theta_s\,\mathrm{d}B_s\Bigg)\Bigg].\tag{3}$$

If φ is a bounded continuous function with compact support, for each $\varepsilon>0$, we can find a $\bar{\varphi}\in C_{b,Lip}(R)$ such that $\sup_{x\in R}|\varphi(x)-\bar{\varphi}(x)|\leqslant\frac{\varepsilon}{2}$. Hence, we have

$$\begin{aligned}\Bigg|\widetilde{E}\big[\varphi(\xi)\big]-\sup_{\theta\in\Theta}E_P\Bigg[\varphi\Bigg(\int_0^1\theta_s\,\mathrm{d}B_s\Bigg)\Bigg]\Bigg|&\leqslant\big|\widetilde{E}\big[\varphi(\xi)\big]-\widetilde{E}\big[\bar{\varphi}(\xi)\big]\big|+\Bigg|\widetilde{E}\big[\bar{\varphi}(\xi)\big]-\sup_{\theta\in\Theta}E_P\Bigg[\bar{\varphi}\Bigg(\int_0^1\theta_s\,\mathrm{d}B_s\Bigg)\Bigg]\Bigg|\\&\quad+\Bigg|\sup_{\theta\in\Theta}E_P\Bigg[\varphi\Bigg(\int_0^1\theta_s\,\mathrm{d}B_s\Bigg)\Bigg]-\sup_{\theta\in\Theta}E_P\Bigg[\bar{\varphi}\Bigg(\int_0^1\theta_s\,\mathrm{d}B_s\Bigg)\Bigg]\Bigg|\leqslant\varepsilon.\end{aligned}$$

Since ε can be arbitrarily small, $\widetilde{E}[\varphi(\xi)]=\sup_{\theta\in\Theta}E_P[\varphi(\int_0^1\theta_s\,\mathrm{d}B_s)]$. $\square$

With the notion of IID under sublinear expectations, Peng shows central limit theorem under sublinear expectations (see Theorem 5.1 in Peng [6]).

Lemma 2.3 *(Central limit theorem under sublinear expectations). Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of IID random variables. We further assume that $\overline{E}[X_1]=\overline{E}[-X_1]=0$. Then the sequence $\{\overline{S}_n\}_{n=1}^{\infty}$ defined by $\overline{S}_n:=\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$ converges in law to ξ, i.e.,*

$$\lim_{n\to\infty}\overline{E}\big[\varphi(\overline{S}_n)\big]=\widetilde{E}\big[\varphi(\xi)\big],$$

for any continuous function φ satisfying linear growth condition, where ξ is a G-normal distribution.

3. Main result

Now we give our main result:

Theorem 3.1 *(Central limit theorem for capacities). Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of IID random variables. We further assume that $\overline{E}[X_1]=\overline{E}[-X_1]=0$. Denote $\overline{S}_n:=\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$. Then*
(1) if y is a point at which $\widetilde{V}$ is continuous, we have

$$\lim_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)=\widetilde{V}(y),$$

(2) if y is a point at which $\tilde{v}$ is continuous, we have

$$\lim_{n\to\infty}\bar{v}(\overline{S}_n\leqslant y)=\tilde{v}(y),$$

where $\widetilde{V}(y)=\sup_{\theta\in\Theta}E_P[I_{\{\int_0^1\theta_s\,\mathrm{d}B_s\leqslant y\}}]$ and $\tilde{v}(y)=\inf_{\theta\in\Theta}E_P[I_{\{\int_0^1\theta_s\,\mathrm{d}B_s\leqslant y\}}]$.

Proof. Suppose that y is a point at which $\widetilde{V}$ is continuous. Let ε be any positive number, and take δ small enough that $\widetilde{V}(y+\delta)-\widetilde{V}(y-\delta)\leqslant\varepsilon$. Construct two bounded continuous functions f, g such that

$$f(x)=1 \quad \text{for } x\leqslant y-\delta, \qquad f(x)=0 \quad \text{for } x\geqslant y, \qquad 0<f(x)\leqslant 1 \quad \text{for } y-\delta<x<y;$$
$$g(x)=1 \quad \text{for } x\leqslant y, \qquad g(x)=0 \quad \text{for } x\geqslant y+\delta, \qquad 0<g(x)\leqslant 1 \quad \text{for } y<x<y+\delta.$$

Then

$$\widetilde{V}(y-\delta)\leqslant\sup_{\theta\in\Theta}E_P\left[f\left(\int_0^1\theta_s\,\mathrm{d}B_s\right)\right]\leqslant\widetilde{V}(y)\leqslant\sup_{\theta\in\Theta}E_P\left[g\left(\int_0^1\theta_s\,\mathrm{d}B_s\right)\right]\leqslant\widetilde{V}(y+\delta), \tag{4}$$

and for each n,

$$\overline{E}\big[f(\overline{S}_n)\big]\leqslant\overline{V}(\overline{S}_n\leqslant y)\leqslant\overline{E}\big[g(\overline{S}_n)\big]. \tag{5}$$

Obviously, f and g have compact supports. By Lemmas 2.2 and 2.3, we have

$$\lim_{n\to\infty}\overline{E}\big[f(\overline{S}_n)\big]=\sup_{\theta\in\Theta}E_P\left[f\left(\int_0^1\theta_s\,\mathrm{d}B_s\right)\right],$$

$$\lim_{n\to\infty}\overline{E}\big[g(\overline{S}_n)\big]=\sup_{\theta\in\Theta}E_P\left[g\left(\int_0^1\theta_s\,\mathrm{d}B_s\right)\right].$$

So that

$$\sup_{\theta\in\Theta}E_P\left[f\left(\int_0^1\theta_s\,\mathrm{d}B_s\right)\right]\leqslant\liminf_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)\leqslant\limsup_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)\leqslant\sup_{\theta\in\Theta}E_P\left[g\left(\int_0^1\theta_s\,\mathrm{d}B_s\right)\right]. \tag{6}$$

Hence

$$\widetilde{V}(y)-\varepsilon\leqslant\liminf_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)\leqslant\limsup_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)\leqslant\widetilde{V}(y)+\varepsilon. \tag{7}$$

Since this is true for every ε, $\lim_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)=\widetilde{V}(y)$. $\square$

In a similar manner as in the above, we can obtain $\lim_{n\to\infty}\bar{v}(\overline{S}_n\leqslant y)=\tilde{v}(y)$.

Remark 3.1. (1) Obviously, $\widetilde{V}$ is an increasing function, then $\widetilde{V}$ is continuous in R except in, at most, countable points. Similarly, $\tilde{v}$ is continuous in R except in, at most, countable points.

(2) In Theorem 3.1, if $\overline{E}[X_1^2]=-\overline{E}[-X_1^2]=\sigma^2>0$, then for each $y\in R$,

$$\lim_{n\to\infty}\overline{V}(\overline{S}_n\leqslant y)=\lim_{n\to\infty}\bar{v}(\overline{S}_n\leqslant y)=\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x.$$

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