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Geometry

Primitive stable representations of geometrically infinite handlebody hyperbolic 3-manifolds

Représentations primitivement stables des variétés hyperboliques géométriquement infinies du bretzel creux

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A R T I C L E I N F O

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A B S T R A C T

In this Note we show that a discrete faithful representation of a free group in $PSL(2,\mathbb{C})$ without parabolics is primitive stable.

R É S U M É

Nous démontrons qu'une représentation discrète, fidèle du groupe libre dans $PSL(2,\mathbb{C})$ sans parabolique est primitivement stable.

Version française abrégée

On dit qu'un élément du groupe libre est primitif s'il appartient à un système de générateurs. On montre qu'une représentation fidèle et discrète sans parabolique du groupe libre dans $PSL(2,\mathbb{C})$ est primitivement stable, c'est-à-dire, les orbites des éléments primitifs dans $\mathbb{H}^3$ sont uniformément quasi-géodésiques. Ce résultat résout la conjecture de Minsky. Pour le cas avec paraboliques, on suppose que chaque composante de la lamination terminale est doublement incompressible.

1. Introduction

Let F be a free group of rank n and Γ a bouquet of n oriented circles realizing F with respect to a fixed generating set $X = \{x_1, \ldots, x_n\}$. Then $\tilde{\Gamma}$ is a Cayley graph of F with respect to X. To every conjugacy class $[w]$ in F is associated a bi-infinite oriented geodesic in Γ named $\bar{w}$, namely the periodic word determined by concatenating infinitely many copies of a cyclically reduced representative of w. An element of F is called *primitive* if it is a member of a free generating set and let $\mathcal{P}$ denote the set consisting of $\bar{w}$ for conjugacy classes $[w]$ of primitive elements, which is $Out(F)$-invariant.

Given a representation $\rho : F \to PSL_2(\mathbb{C})$ and a base point $o \in \mathbb{H}^3$, there is a unique ρ-equivariant map $\tau_{\rho,o} : \tilde{\Gamma} \to \mathbb{H}^3$ mapping the origin e of $\tilde{\Gamma}$ to o and mapping each edge to a geodesic segment. Any $\bar{w}$ is represented by an F-invariant family of leaves in $\tilde{\Gamma}$, which map to a family of broken geodesic paths in $\mathbb{H}^3$.

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A representation $\rho : F \to PSL_2(\mathbb{C})$ is *primitive stable* if there are constants K, δ and a base point $o \in \mathbb{H}^3$ such that $\tau_{\rho,o}$ takes all leaves of $\mathcal{P}$ to (K, δ)-quasi geodesics in $\mathbb{H}^3$. For each $\bar{w}$, we will choose a specified lift $\tilde{w}$ passing through $e \in \tilde{\Gamma}$. If each $\tilde{w}$ is mapped by $\tau_{\rho,o}$ to a uniform quasigeodesic in $\mathbb{H}^3$ then ρ will be primitive stable.

Minsky [10] showed that

(i) If ρ is Schottky then it is primitive stable.
(ii) The set of primitive stable representations up to conjugacy, $\mathcal{PS}$, is open.
(iii) If ρ is primitive stable then, for every free factor A of F, $\rho|_A$ is Schottky.

In the same paper, he conjectured that

(i) Every discrete faithful representation of F without parabolics is primitive stable.
(ii) A discrete faithful representation of F is primitive stable if and only if every component of its ending lamination is blocking.

Since every geometrically finite representation of F without parabolics is Schottky and every Schottky group is primitive stable [10], the following theorem settles down the first conjecture:

Theorem 1. *Let $M = \mathbb{H}^3/\rho(F)$ be a geometrically infinite hyperbolic manifold without parabolics. Then ρ is primitive stable.*

We also answer the second conjecture partially.

Theorem 2. *Suppose ρ is a geometrically infinite discrete faithful representation with parabolics with an ending lamination $\lambda = \bigcup \lambda_i$ together with parabolic loci. If $M = \mathbb{H}^3/\rho(F)$ has a non-cuspidal part $M_0 = H \cup E_i$ where $E_i = S_i \times [0, \infty)$ corresponding to an incompressible S_i is geometrically finite, and where E_i corresponding to a compressible S_i has a doubly incompressible ending lamination λ_i, then ρ is primitive stable.*

2. Proof of the main theorem

Let H be a genus n handlebody. A measured lamination λ on ∂H is *doubly incompressible* if for any essential disc or annulus A, $i(\partial A, \lambda) > 0$ where i denotes the intersection form. The set of doubly incompressible measured laminations is strictly bigger than the Masur domain [8]. Let $\triangle = \{\delta_1, \ldots, \delta_n\}$ be a system of compressing disks on H along which one can cut H into a 3-ball. A free generating set of $\pi_1(H) = F_n = F$ is dual to such a system. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the free generating set dual to $\triangle$. $Wh(g, X)$, the Whitehead graph of a cyclically reduced primitive word g with respect to a generating set of F, is defined as follows [14,15,13]. $Wh(g, X)$ is a graph with $2n$ vertices $X \cup X^{-1} = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\}$ and two vertices x, y^{-1} is joined by an edge from x to y^{-1} whenever the string xy appears in g or in a cyclic permutation of g.

Lemma 2.1 *(Whitehead). Let g be a cyclically reduced word in a free group F, and let X be a fixed generating set. If $Wh(g, X)$ is connected and has no cutpoint, then g is not primitive.*

Given a doubly incompressible measured lamination λ, we can find a system of compressing disks $\triangle$ which cut H into a 3-ball so that every arc of $\lambda \backslash \triangle$ is in *tight position* with respect to $\triangle$. For details, see [12,9,10]. When we cut ∂H along $\triangle$, we get $2n$ boundary circles, each labeled by δ_i^+, δ_i^- and $Wh(\lambda, \triangle)$ can be defined as the graph whose vertices and edges are $2n$ boundary circles and arcs in $\lambda \backslash \triangle$ respectively. It is not difficult to see that $Wh(g, X)$ is equivalent to $Wh(g, \triangle)$ for a cyclically reduced word g if $\triangle$ is a dual system to X. The following lemma is essentially due to Otal [12], see also [9,10]:

Lemma 2.2. *Let λ be a doubly incompressible measured lamination. Then there is a generating set with the dual disk system $\triangle$ so that $Wh(\lambda, \triangle)$ is connected and has no cutpoints.*

Let $\rho : F \to PSL(2, \mathbb{C})$ be a geometrically infinite discrete faithful representation without parabolics and $M = \mathbb{H}^3/\rho(F)$. Then by tameness theorem [1,3], $M = H \cup E$ where H is the compact genus n handlebody and E is the compressible end homeomorphic to $\partial H \times [0, \infty)$. In this case, the existence of the Cannon–Thurston map for free groups, and its main property can be stated as follows:

Theorem 2.3. *(See [11,5].) Let $\tilde{H}$ denote the inverse image of H in $\mathbb{H}^3$ and let $\hat{H} = \tilde{H} \cup \partial \tilde{H}$ where $\partial \tilde{H}$ is the Gromov boundary. Define $\tilde{M}, \hat{M}$ similarly. Then the inclusion $\tilde{i} : \tilde{H} \to \tilde{M}$ extends continuously to a map $\hat{i} : \hat{H} \to \hat{M}$. Let $\hat{i}(a) = \hat{i}(b)$ for a, b two distinct points that are identified by the Cannon–Thurston map. Then a, b are either ideal end-points of a leaf of the ending lamination or ideal boundary points of a complementary ideal polygon.*

Let us choose a hyperbolic metric on ∂H and let γ be the geodesic homotopic to the projection to ∂H of the unique bi-infinite path joining a and b as in the above theorem. Note that an ending lamination λ of M consists of just one minimal component so every leaf is dense and any isolated bi-infinite geodesic spiraling to λ has the minimal component in its closure. Thus the closure of γ in ∂H contains λ. Furthermore, by Canary [4], λ is in the Masur domain so is doubly incompressible. Here we give the proof of our main theorem.

Proof of Theorem 1. Recall that $M = \mathbb{H}^3/\rho(F) = H \cup (\partial H \times [0, \infty))$. Regard each cyclically reduced primitive word w as a covering transformation of $\tilde{\Gamma}$ and let $\tilde{w}$ be the unique bi-infinite path in $\tilde{\Gamma}$ passing through $w^k(e)$ for all $k \in \mathbb{Z}$. Then its image under $\tau_{\rho,o}$ passes through o. Suppose ρ is not primitive stable and let γ_{w_n} be the hyperbolic bi-infinite geodesic which has the same end-points as the broken geodesic $\tau_{\rho,o}(\tilde{w}_n)$ with respect to a chosen hyperbolic metric on ∂H. We claim that we can choose a sequence of cyclically reduced primitive words w_n such that γ_{w_n} leaves every compact set in $\mathbb{H}^3$ as $n \to \infty$. □

Proof of the claim. Note that when we identify the core curves of H with Γ, if every geodesic is contained in a uniformly thickened Γ in $M = \mathbb{H}^3/\rho(F)$, then ρ is primitive stable, see Lemma 3.2. in [10]. Since ρ is not primitive stable, there exists a sequence of cyclically reduced primitive words $\{w_n\}$ and a sequence of positive numbers $\{\epsilon_n\}$ such that the projection of γ_{w_n} is not contained in ϵ_n-neighborhood of the core curves of H where $\epsilon_n \to \infty$. Thus γ_{w_n} is not contained in ϵ_n-neighborhood of $\tau_{\rho,o}(\tilde{\Gamma})$ in $\mathbb{H}^3$ and not in ϵ_n-neighborhood of $\tau_{\rho,o}(\tilde{w}_n)$ either. In particular, we can choose a vertex of $\tau_{\rho,o}(\tilde{w}_n)$ whose minimal distance from γ_{w_n} is larger than ϵ_n. Moreover, we can shift w_n's so that the specified vertex is the base point o as follows.

Let the vertex be $\rho(w_n^i v_n)o$ where $w_n = g_1 g_2 \cdots g_k$ and $v_n = g_1 \cdots g_l$ for $l < k$ and $i \in \mathbb{Z}$. Assuming $d_{\mathbb{H}^3}(\rho(w_n^i v_n)o, \gamma_{w_n}) > \epsilon_n$ and noting that γ_{w_n} is the axis of the loxodromic isometry $\rho(w_n)$, we get

$$d_{\mathbb{H}^3}\big(\rho\big(w_n^i v_n\big)o, \gamma_{w_n}\big) = d_{\mathbb{H}^3}\big(\rho(v_n)o, \gamma_{w_n}\big) = d_{\mathbb{H}^3}\big(o, \rho(v_n)^{-1}\gamma_{w_n}\big)$$

and

$$\rho(v_n)^{-1}\gamma_{w_n} = \gamma_{v_n^{-1} w_n v_n}.$$

Then $v_n^{-1} w_n v_n$ is a shifted word so it is also primitive. Finally we get

$$d_{\mathbb{H}^3}(o, \gamma_{v_n^{-1} w_n v_n}) > \epsilon_n.$$

Thus $\gamma_{v_n^{-1} w_n v_n}$ has to leave every compact set in $\mathbb{H}^3$ and $\{v_n^{-1} w_n v_n\}$ is our required sequence. This proves the claim. Denote this sequence again by $\{w_n\}$ by using a slight abuse of notation. We further reduce $\{w_n\}$ to a subsequence such that for all $i > 0$, $w_{i+1} = w_i g_1 g_2 \cdots g_k$ for some $k > 0$ where $g_j \in X \cup X^{-1}$. This is a variant of Cantor diagonal process mentioned in [6].

Now let $\tilde{w}_\infty$ be the limit of $\tilde{w}_n$. Since γ_{w_n} leaves every compact set of $\mathbb{H}^3$ as $n \to \infty$, the Cannon–Thurston map $\hat{i}$ maps the end-points of $\tilde{w}_\infty$ to a point $p \in \partial\mathbb{H}^3$. Let γ_n be the geodesic representing w_n on the boundary of the handlebody and let γ_∞ be their Hausdorff limit. Here we appeal to Theorem 2.3, which implies that the closure of γ_∞ must contain the ending lamination λ of M. Since $Wh(\lambda, \triangle)$ is connected and has no cutpoints with respect to some $\triangle$ by Lemma 2.2, the same is true for $Wh(\gamma_n, \triangle)$ for large n. But for any primitive word w_n, this is impossible by Whitehead Lemma 2.1. □

The proof of Theorem 2 can be done analogously. Call a lamination λ *blocking* with respect to $\triangle$ if it is in tight position and there exists some k such that every length k subword of the infinite word determined by a leaf of λ does not appear in a cyclically reduced primitive word. Lemma 4.6 in [10] can be generalized as follows:

Lemma 2.4. *A connected doubly incompressible lamination λ on the boundary of a handlebody is blocking with respect to some generating set.*

Proof of Theorem 2. If $M_i = H_i \cup E_i$ is the covering manifold corresponding to $\pi_1(S_i)$, then the end E_i is bilipschitz homeomorphic to an end of a simply degenerate hyperbolic manifold homeomorphic to $S_i \times \mathbb{R}$ [2]. Then the rest of the proof is the combination of [11] and [5]. See [7] for details. □

References

[1] I. Agol, Tameness of hyperbolic 3-manifolds, arXiv:math.GT/0405568, 2004, preprint.
[2] B. Bowditch, Geometric model for hyperbolic manifolds, Southhampton, 2005, preprint.
[3] D. Calegari, D. Gabai, Shrinkwrapping and the taming of hyperbolic 3-manifolds, J. Amer. Math. Soc. 19 (2) (2006) 385–446.
[4] R.D. Canary, Ends of hyperbolic 3-manifolds, J. Amer. Math. Soc. 6 (1) (1993) 1–35.
[5] S. Das, M. Mj, Addendum to ending laminations and Cannon–Thurston maps: Parabolics, arXiv:1002.2090, 2010, preprint.
[6] W.J. Floyd, Group completions and limit sets of Kleinian groups, Invent. Math. 57 (1980) 205–218.
[7] W. Jeon, I. Kim, On primitive stable representations of geometrically infinite handlebody hyperbolic 3-manifolds, arXiv:1003.2055, 2010, preprint.

[8] I. Kim, Divergent sequences of function groups, Differential Geom. Appl. 26 (6) (2008) 645–655.
[9] I. Kim, C. Lecuire, K. Ohshika, Convergence of freely decomposable Kleinian groups, arXiv:0708.3266, 2004, preprint.
[10] Y. Minsky, On dynamics of $Out(F_n)$ on $PSL_2(\mathbb{C})$ characters, arXiv:0906.3491, 2009, preprint.
[11] M. Mj, Cannon–Thurston maps for Kleinian groups, arXiv:1002.0996, 2010, preprint.
[12] J.-P. Otal, Courants géodésiques et produits libres, Thèse d'Etat, Université de Paris-Sud, Orsay, 1988.
[13] J.R. Stallings, Whitehead graphs on handlebodies, 1996, preprint.
[14] J.H.C. Whitehead, On certain sets of elements in a free group, Proc. London Math. Soc. 41 (1936) 48–56.
[15] J.H.C. Whitehead, On equivalent sets of elements in a free group, Ann. of Math. 37 (1936) 780–782.