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Graph eigenfunctions and quantum unique ergodicity

Fonctions propres de graphes et l'unique ergodicité quantique

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ABSTRACT

We apply the techniques of Brooks and Lindenstrauss (2010) [5] to study joint eigenfunctions of the Laplacian and one Hecke operator on compact congruence surfaces, and joint eigenfunctions of the two partial Laplacians on compact quotients of $\mathbb{H} \times \mathbb{H}$. In both cases, we show that quantum limit measures of such sequences of eigenfunctions carry positive entropy on almost every ergodic component. Together with the work of Lindenstrauss (2006) [9], this implies Quantum Unique Ergodicity for such functions.

RÉSUMÉ

On applique les techniques de Brooks et Lindenstrauss (2010) [5] pour étudier fonctions propres jointes du laplacien et d'un opérateur Hecke sur des surfaces compactes de congruence, et les fonctions propres jointes de deux laplaciens partiels sur les quotients compacts de $\mathbb{H} \times \mathbb{H}$. Dans les deux cas, on montre entropie strictement positive sur presque toutes les composantes ergodiques des limites quantiques. De plus, les travaux de Lindenstrauss (2006) [9] ce implique Unique Ergodicité Quantique pour ces fonctions.

Version française abrégée

Notre premier résultat concerne certaines surfaces compactes arithmétiques $\Gamma\backslash\mathbb{H}$ de type congruence. On peut considérer des surfaces plus générales, mais pour simplifier, on se limite à la situation suivante : Soit H une algèbre de quaternions à division sur $\mathbb{Q}$, scindée sur (R), et R un ordre dans H. Fixons un isomorphisme $\Psi : H(\mathbb{R}) \cong Mat_2(\mathbb{R})$. Si la norme $n(\alpha)$ de $\alpha \in R$ est positive, on écrit $\underline{\alpha} = n(\alpha)^{-1/2}\Psi(\alpha) \in SL_2(\mathbb{R})$. On note Γ l'image par Ψ du sous-groupe des éléments de R ayant de norme unitaire. Le sous-groupe Γ est discret et co-compact dans $SL_2(\mathbb{R})$, et le quotient $X = \Gamma\backslash SL_2(\mathbb{R})$ est donc un recouvrement à deux feuillets du fibré unitaire cotangent d'une surface hyperbolique compacte $M = \Gamma\backslash\mathbb{H}$.

On écrit $R(m)$ l'ensemble des éléments de R de norme m, et on définit l'opérateur de Hecke :

$$T_m : f(x) \mapsto \frac{1}{\sqrt{m}} \sum_{\alpha \in R(1)\backslash R(m)} f(\underline{\alpha} x),$$

l'opérateur qui associe à f sa moyenne sur les points de Hecke, $T_m(x) = \{\underline{\alpha} x\colon\ \alpha \in R(1)\backslash R(m)\}$. On s'intéresse au cas des puissances d'un nombre premier p, $m = p^k$. L'opérateur T_{p^k} est un polynôme en T_p ; donc en particulier, les fonctions

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propres de T_p sont des fonctions propres de tous les T_{p^k}. Pour presque tous les nombres premiers, les points de $T_{p^k}(x)$ forment un arbre régulier ; on peut toujours supposer que p est un « bon » nombre premier. Nous désignons par S_{p^k} la sphère de rayon k dans cet arbre, donnée par les points de Hecke correspondant aux éléments primitifs de R de norme p^k.

Pour toute fonction propre ϕ_j du laplacien Δ sur M, normalisée par $\|\phi_j\|_2 = 1$, nous avons la construction d'une mesure $\mu_j = |\Phi_j|^2\, dVol$ sur S^*M (considérée comme une mesure sur $\Gamma\backslash SL(2,\mathbb{R})$) appelée **relèvement microlocal** de ϕ_j, elle est asymptotiquement invariante par rapport au flot géodésique. On va utiliser la variante de Wolpert [13] de cette construction, utilisée dans [8,9], où $\mu_j = |\Phi_j|^2\, dVol$ pour une fonction $\Phi_j \in L^2(S^*M)$ appropriée dans la même représentation indécomposable de $SL_2(\mathbb{R})$ sur $L^2(S^*M)$ que ϕ_j. Puisque Δ commute avec tous les T_p, on peut considérer des suites $\{\phi_j\}$ des fonctions propres jointes, et chaque Φ_j est donc aussi une fonction propre de T_p (avec la même valeur propre que ϕ_j).

Théorème 1. *Soit p un nombre premier (n'appartenant pas à l'ensemble fini des nombres premiers mauvais pour M), et soit* $\{\phi_j\}_{j=1}^{\infty}$ *une suite de fonctions propres jointes normalisées de* Δ *et* T_p *sur M. Alors, tout point d'accumulation faible** μ *des* $|\Phi_j|^2\, dVol$ *a une entropie strictement positive sur presque toutes les composantes ergodiques.*

Noter que Anantharaman [1] a montré, pour toutes les variétés compactes à courbure négative, et sans aucun opérateur de Hecke, que la limite quantique a une entropie strictement positive, et cela a été encore accentué dans son article joint avec Nonnenmacher [3] (voir aussi [2]). Donc la contribution du Théorème 1 est qu'il donne des informations sur *presque toutes les composantes ergodiques* d'une limite quantique. De plus les résultats de la classification des mesures de [9], impliquent :

Corollaire 1. *Soit* $\{\phi_j\}$ *comme ci-dessus une suite de fonctions propres jointes de* Δ *et* T_p*. Alors, la suite* μ_j *converge vers la mesure de Liouville sur* S^*M*.*

Dans [9], on a montré que si la suite $\{\phi_j\}$ se compose de fonctions propres jointes de Δ et de *tous* les opérateurs de Hecke, alors les μ_j convergent vers la mesure de Liouville. L'hypothèse selon laquelle ϕ_j est une fonction propre de tous les opérateurs de Hecke a été utilisée seulement pour établir, dans les travaux de Bourgain et de second auteur, que μ a une entropie strictement positive sur presque toutes les composantes ergodiques [4]. Il résulte que le Théorème 1 est, au moins formellement, un renforcement du résultat de [9]. Il convient de noter que si les multiplicités dans le spectre du laplacien sont bornées uniformément, [9] implique immédiatement Q.U.E. pour toute suite de fonctions propres du laplacien, sans l'hypothèse supplémentaire d'invariance par rapport aux opérateurs de Hecke, notre résultat serait alors une conséquence de celui-ci. Toutefois, on sait très peu de choses sur ces multiplicités.

Nos méthodes s'appliquent aussi au cas de $M = \Gamma\backslash\mathbb{H}\times\mathbb{H}$, où Γ est co-compact et irréductible dans $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. Ici, on suppose aucune structure arithmétique,[2] mais plutôt, on suppose que la suite $\{\phi_j\}$ est composée de fonctions propres jointes de deux laplaciens partiels, chacun sur la copie respective de $\mathbb{H}$ (i.e., le ϕ_j sont des fonctions propres jointes de l'algèbre commutative d'opérateurs différentiels $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$-invariants). Puisque le laplacien sur M est la somme des deux laplaciens partiels, il résulte qu'au moins une des deux suites de valeurs propres partielles vont à l'infini (après passage à une sous-suite, si nécessaire).

Sans perte de généralité, on suppose que les valeurs propres de la première coordonnée va à l'infini. En appliquant notre analyse à l'action de $SL(2,\mathbb{R})$ sur la deuxième coordonnée—qui commute avec l'action sur la première coordonnée—on peut démontrer que la limite quantique d'une telle suite a une entropie strictement positive sur presque toutes les composantes ergodiques (par rapport à l'action du sous-groupe diagonal de $SL(2,\mathbb{R})$ sur la première coordonnée). Ainsi, le résultat de [9] montre que $|\phi_j|^2\, dVol$ converge vers la mesure riemannienne $dVol$ sur M. L'argument est analogue à celui présenté ci-dessous dans le cas arithmétique du Théorème 1.

1. Introduction

The purpose of this Note is to point out an application of the techniques developed in the context of our work [5] on eigenfunctions of large graphs to some Quantum Unique Ergodicity problems.

Our first result concerns certain compact hyperbolic surfaces $\Gamma\backslash\mathbb{H}$ of arithmetic congruence type. One can consider more general Γ, but for concreteness and simplicity we restrict to the following situation. Let H be a quaternion division algebra over $\mathbb{Q}$, split over $\mathbb{R}$, and R an order in H. Fix an isomorphism $\Psi : H(\mathbb{R}) \cong \mathrm{Mat}_2(\mathbb{R})$. For $\alpha \in R$ of positive norm $n(\alpha)$, we write $\underline{\alpha} = n(\alpha)^{-1/2}\Psi(\alpha) \in SL_2(\mathbb{R})$. Set Γ to be the image under Ψ of the subgroup of norm 1 elements of R. As is well known, Γ is discrete and co-compact in $SL_2(\mathbb{R})$, and so the quotient $X = \Gamma\backslash SL_2(\mathbb{R})$ is a 2-to-1 cover of the unit cotangent bundle of a compact hyperbolic surface $M = \Gamma\backslash\mathbb{H}$.

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[2] Le Théorème de Arithméticité de Margulis implique que ces Γ sont nécessairement arithmétique, mais on ne sait pas si elles nécessairement de type congruence (ce qui est nécessaire à l'existence d'opérateurs de Hecke avec de bonnes propriétés).

Write $R(m)$ for the set of elements of R of norm m, and define the Hecke operator,

$$T_m : f(x) \mapsto \frac{1}{\sqrt{m}} \sum_{\alpha \in R(1)\backslash R(m)} f(\underline{\alpha}x),$$

as the operator averaging over the Hecke points $T_m(x) = \{\underline{\alpha}x\colon\ \alpha \in R(1)\backslash R(m)\}$. We will be interested in the case where $m = p^k$ are powers of a fixed prime p. It is well known that T_{p^k} is a polynomial in T_p; so in particular, eigenfunctions of T_p are eigenfunctions of all T_{p^k}. For all but finitely many primes, the points $T_{p^k}(x)$ form a $p+1$-regular tree as k runs from 0 to ∞; we will always assume that p is such a prime. We denote by S_{p^k} the sphere of radius k in this tree, given by Hecke points corresponding to the primitive elements of R of norm p^k.

For any eigenfunction ϕ_j of the Laplacian Δ on M, normalized by $\|\phi_j\|_2 = 1$, one can construct a measure μ_j on S^*M (which we view as a measure on the double cover $\Gamma\backslash SL_2(\mathbb{R})$) called the **microlocal lift** of ϕ_j which is asymptotically invariant under the geodesic flow as the Laplace eigenvalue of ϕ_j tends to infinity. We shall use the variant of this construction used in [8,9], due to Wolpert [13], where $\mu_j = |\Phi_j|^2 dVol$ for suitably chosen $\Phi_j \in L^2(S^*M)$ in the irreducible representation of $SL_2(\mathbb{R})$ on $\Gamma\backslash SL_2(\mathbb{R})$ generated by translates of ϕ_j. The construction satisfies that Φ_j is an eigenfunction of T_p when ϕ_j is. Since Δ commutes with all of the T_p, we may consider sequences $\{\phi_j\}$ of joint eigenfunctions, whereby each Φ_j is also an eigenfunction of T_p (with the same eigenvalue as ϕ_j).

Theorem 1. *Let p be a prime (outside the finite set of bad primes for M), and let $\{\phi_j\}_{j=1}^{\infty}$ be a sequence of L^2-normalized joint eigenfunctions of Δ and T_p on M. Then any weak-* limit point μ of the microlocal lifts μ_j has positive entropy on almost every ergodic component.*

Note that even without any Hecke operators, Anantharaman [1] has shown (for general negatively curved compact manifolds) that any quantum limit has positive entropy, and this has been further sharpened in her joint work with Nonnenmacher [3] (see also [2]). Hence the point of Theorem 1 is that it gives information on *almost all ergodic components* of a quantum limit. In view of the measure classification results of [9], this implies the following:

Corollary 1. *Let $\{\phi_j\}$ as above be a sequence of joint eigenfunctions of Δ and T_p. Then the sequence μ_j converges weak-* to Liouville measure on S^*M.*

In [9], it was shown that if the sequence $\{\phi_j\}$ consists of joint eigenfunctions of Δ and *all* Hecke operators, then the μ_j converge to Liouville measure. The assumption that ϕ_j is an eigenfunction of $\underline{\text{all}}$ Hecke operators was used only to establish, through the work of Bourgain and the second named author [4], that μ has positive entropy on a.e. ergodic component. Hence Theorem 1 gives, at least formally, a strengthening of the result of [9]. It should be noted that if the multiplicities in the Laplace spectrum are uniformly bounded—as is conjectured to be true in this setting—then [9] immediately implies QUE (Quantum Unique Ergodicity) for any sequence of Laplace eigenfunctions, without making an additional assumption of Hecke invariance, and our result would be absorbed therein. However, very little is known about these multiplicities at the present time.

Our methods also apply to the case of $M = \Gamma\backslash\mathbb{H}\times\mathbb{H}$, with Γ a co-compact, irreducible lattice in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. Here we do not assume any arithmetic structure[3]; instead, we take the sequence $\{\phi_j\}$ to consist of joint eigenfunctions of the two partial Laplacians, each on the respective copy of $\mathbb{H}$ (equivalently, the ϕ_j are joint eigenfunctions of the full commutative algebra of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$-invariant differential operators). The Laplacian on M is the sum of the two partial Laplacians, and so the large eigenvalue limit for the Laplacian entails at least one of the two partial eigenvalues going to infinity (after passing to a subsequence, if necessary).

Without loss of generality, we assume that the eigenvalues in the first coordinate are tending to infinity. By usual semiclassical arguments (see [8]), this means that the microlocal lift to $\Gamma\backslash SL(2,\mathbb{R})\times\mathbb{H}$ becomes invariant under the action of the diagonal subgroup A of $SL(2,\mathbb{R})$ acting on the first coordinate. By applying our analysis to the action of $SL(2,\mathbb{R})$ on the second coordinate—which commutes with the action on the first coordinate—"in place" of the Hecke operator T_p, we are able to prove that any quantum limit of such a sequence must also carry positive entropy on a.e. ergodic component (with respect to the A-action on the first coordinate). Thus, the result of [9] again applies to show that $|\phi_j|^2\,dVol$ converges weak-* to the Riemannian measure $dVol$ on M. The argument is analogous to the one presented here for the rank-one arithmetic case in Theorem 1, and we intend to supply complete proofs of both results in a forthcoming paper.

The results of [4] were generalized by Silberman and Venkatesh [10,11] who, using the measure classification results of [6], were able to extend the QUE results of [9] to quotients of more general symmetric spaces (they also had to develop an appropriate microlocal lift). It is likely possible to extend the techniques of this paper to their context.

Regarding finite volume arithmetic surfaces such as $SL(2,\mathbb{Z})\backslash\mathbb{H}$, it was shown in [9] that any quantum limit has to be a scalar multiple of the Liouville measure, though not necessarily with the right scalar, and similar results can be provided by

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[3] By Margulis' Arithmeticity Theorem such lattices are necessarily arithmetic, though it is not known if they are necessarily of congruence type (which is necessary for the existence of Hecke operators with good properties).

our techniques using a single Hecke operator. Recently, Soundararajan [12] has given an elegant argument that settles this escape of mass problem and (in view of the results of [9]) shows that the only quantum limit is the normalized Liouville measure. An interesting open question is whether our p-adic wave equation techniques can be used to rule out escape of mass using a single Hecke operator. We also mention that Holowinsky and Soundararajan [7] have recently developed an alternative approach to establishing Arithmetic Quantum Unique Ergodicity for joint eigenfunctions of all Hecke operators. This approach *requires* a cusp, and is only applicable in cases where the Ramanujan Conjecture holds; this conjecture is open for the Hecke–Maass forms, but has been established by Deligne for holomorphic cusp forms—a case which our approach does not handle.

2. The propagation lemma

The following lemma proved along the lines of [5] is central to our approach:

Lemma 1. *Let $\eta > 0$. For any sufficiently large $N \in \mathbb{N}$ (depending on η), and any T_p eigenfunction Φ_j, there exists a convolution operator K_N on S^*M satisfying*:

- *$K_N(\delta_x)$ is supported on the union of Hecke points $y \in T_{p^j}(x)$ up to distance $j \leqslant N$ in the Hecke tree, and is constant on the spheres $S_{p^j}(x)$.*
- *K_N has matrix coefficients bounded by $O(p^{-N\delta})$, in the sense that for any $x \in S^*M$,*

$$\left|K_N(f)(x)\right| \lesssim p^{-N\delta} \sum_{j=0}^{N} \sum_{y \in S_{p^j}(x)} f(y).$$

- *Any T_p eigenfunction is also an eigenfunction of K_N, of eigenvalue $\geqslant -1$. Moreover, Φ_j has K_N-eigenvalue $> \eta^{-1}$.*

Lemma 1 is based on the well known connection between Hecke operators and Chebyshev polynomials. A way to derive these which we have found helpful is via the following p-adic wave equation for functions on $\mathcal{G}$

$$\Phi_{n+1} = \frac{1}{2} T_p \Phi_n - \left(1 - \frac{T_p^2}{4}\right) \Psi_n, \qquad \Psi_{n+1} = \frac{1}{2} T_p \Psi_n + \Phi_n,$$

which is a discrete analog of the non-Euclidean wave equation (more precisely, of the unit time propagation map for the wave equation) on $\mathbb{H}$. For initial data $(\Phi_0, \Psi_0) \in L^2(\mathcal{G}) \times L^2(\mathcal{G})$, the solution to this equation is given by the sequence:

$$\Phi_n = P_n\left[\frac{1}{2} T_p\right] \Phi_0 - \left(1 - \frac{T_p^2}{4}\right) Q_{n-1}\left[\frac{1}{2} T_p\right] \Psi_0, \qquad \Psi_n = P_n\left[\frac{1}{2} T_p\right] \Psi_0 + Q_{n-1}\left[\frac{1}{2} T_p\right] \Phi_0,$$

where P and Q are Chebyshev polynomials of the first and second kinds, respectively, given by

$$P_n(\cos\theta) = \cos n\theta, \qquad Q_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}.$$

This can be proved directly by induction, using the recursive properties of the Chebyshev polynomials.

Suppose we take initial data $(\delta_0, 0)$. The solution to the p-adic wave equation is then $\{(P_n[\frac{1}{2} T_p]\delta_0, Q_{n-1}[\frac{1}{2} T_p]\delta_0)\}$. On the other hand, we can compute the explicit solution inductively; looking at the first coordinate, we get the following "Propagation Lemma" on the tree:

Lemma 2. *Let δ_0 be the delta function at 0 in the $p + 1$-regular tree. Then for n even, we have*

$$P_n\left[\frac{1}{2} T_p\right] \delta_0(x) = \begin{cases} 0, & |x| \text{ odd or } |x| > n, \\ \frac{1-p}{2p^{n/2}}, & |x| < n \text{ and } |x| \text{ even}, \\ \frac{1}{2p^{n/2}}, & |x| = n. \end{cases}$$

In particular, we have:

$$P_n\left[\frac{1}{2} T_p\right] \delta_0(x) \lesssim p^{-n/2}.$$

We now have a description of the p-adic wave propagation in both spectral and spacial terms, which we will use to construct our desired radial kernel K_N on the Hecke tree. As is already evident in Lemma 2, it will be convenient to write the T_p-eigenvalues as $2\cos(\theta)$, where

- The tempered spectrum is parametrized by $\theta \in [0, \pi]$.
- The positive part of the untempered spectrum has $i\theta \in (0, \log \sqrt{p})$.
- The negative part of the untempered spectrum has $i\theta + \pi \in (0, \log \sqrt{p})$.

Consider first the case where Φ_j has T_p-eigenvalue 2, or $\theta = 0$. Denoting the Fejér kernel of order M by F_M, we set the spherical transform of K_N to be $h_{K_N}(\theta) = F_M(q\theta) - 1$. Now $F_M(0) = M$ and F_M is non-negative, so the third condition of Lemma 1 is satisfied on the tempered spectrum, as long as $M > \eta^{-1} + 1$. Moreover, we can write

$$F_M(q\theta) - 1 = \sum_{j=1}^{M} \frac{2(M-j)}{M} \cos(jq\theta)$$

and observing that $\cos(jq\theta) > \cos(0)$ on the entire untempered spectrum as long as q is even, we see that the third condition holds on the full spectrum. We also observe that

$$\begin{aligned}\|K_N\|_{L^1(\mathcal{G}) \to L^\infty(\mathcal{G})} &\leqslant \sum_{j=1}^{M} 2 \left\| P_{jq}\left[\frac{1}{2} T_p\right] \right\|_{L^1(\mathcal{G}) \to L^\infty(\mathcal{G})} \\ &\lesssim \sum_{j=1}^{M} p^{-jq/2} \lesssim p^{-q/2}\end{aligned}$$

which satisfies the second condition of Lemma 1, as long as $q \geqslant 2N\delta$. Moreover, since each $P_{jq}[\frac{1}{2}T_p]\delta_0$ vanishes outside the ball of radius Mq, the first condition is satisfied once $Mq \leqslant N$. So we may take $M = \lceil \eta^{-1} \rceil + 1$, and $q = 2\lfloor N/2M \rfloor$, which yields $\delta = \lfloor q/2N \rfloor \gtrsim \eta$. The same kernel also works for $\theta = \pi$ or untempered θ.

We must now consider $\theta \in (0, \pi)$. By Dirichlet's Theorem, we may find a $q' < N\eta$ such that $q'\theta$ is close to 0; so close, in fact, that we may take as large an even multiple q of q' as necessary to insure that, for a suitable constant c, we have

$$cN\eta^2 < q < 2N\eta$$

while $q\theta$ is still close enough to 0 that we have $F_{2M}(q\theta) > \eta^{-1} + 1$. Thus in this case we can take $\delta \gtrsim c\eta^2$, and Lemma 1 follows.

3. Proof of the main result

The following estimate can be derived using the techniques of [4], specifically Lemmas 3.1 and 3.3 there (much more general statements of this type by Silberman and Venkatesh can be found in [11]).

Lemma 3. *For τ fixed but small enough, there exists a constant c (depending only on τ), such that for any $x, z \in X$, and any $\epsilon < cp^{-2N}$, the tube $zB(\epsilon, \tau)$ contains at most $O(N)$ of the Hecke points*

$$\{\underline{\alpha}x\colon\ \alpha \in T_{p^j}(x) \text{ for some } j \leqslant N\}.$$

Sketch of proof of Theorem 1. Take any sufficiently fine partition $\mathcal{P}$ of S^*M, and consider its refinement under the time one geodesic flow. Any partition element of the refinement is contained in a union of $O_c(1)$ tubes of the form $xB(cp^{-2N}, \tau)$ for some $x \in S^*M$. We assume that c is sufficiently small, according to Lemma 3, independent of η and N.

Take a collection $\{E_1, E_2, \ldots, E_K\}$ of distinct partition elements of the $\lfloor 2N \log p \rfloor$-th refinement of $\mathcal{P}$, of cardinality K, and set $\mathcal{E} = \bigcup_{k=1}^{K} E_k$ to be their union. Let 1_{E_k} denote the characteristic function of each E_k, and similarly $1_{\mathcal{E}} = \sum_{E_k \subset \mathcal{E}} 1_{E_k}$. To each E_k we associate, as above, $O_c(1)$ tubes $B_{k,l} = x_{k,l}B(cp^{-2N}, \tau)$ whose union contains E_k.

Now assume that $\mu(\mathcal{E}) > \eta$; by definition, this implies that there exists a j such that $\mu_j(\mathcal{E}) = \|\Phi_j 1_{\mathcal{E}}\|_2^2 > \eta$ as well. Consider the correlation $\langle K_N(\Phi_j 1_{\mathcal{E}}), \Phi_j 1_{\mathcal{E}} \rangle$. We will estimate this in two different ways. First, since K_N has small matrix coefficients, we have

$$\begin{aligned}\langle K_N(\Phi_j 1_{\mathcal{E}}), \Phi_j 1_{\mathcal{E}} \rangle_{L^2(S^*M)} &= \sum_{k=1}^{K} \langle K_N(\Phi_j 1_{\mathcal{E}}), \Phi_j 1_{E_k} \rangle_{L^2(E_k)} \\ &\leqslant \sum_{k=1}^{K} \|K_N(\Phi_j 1_{\mathcal{E}})\|_{L^2(E_k)} \|\Phi_j 1_{E_k}\|_2 \leqslant \max_{1 \leqslant k \leqslant K} \|K_N(\Phi_j 1_{\mathcal{E}})\|_{L^2(E_k)} \sum_{k=1}^{K} \|\Phi_j 1_{E_k}\|_2 \\ &\leqslant \sum_{i=1}^{K} \max_{1 \leqslant k \leqslant K} \|K_N(\Phi_j 1_{E_i})\|_{L^2(E_k)} \sum_{k=1}^{K} \|\Phi_j 1_{E_k}\|_2 \lesssim \sum_{i=1}^{K} \max_{1 \leqslant k \leqslant K} N p^{-\delta N} \|\Phi_j 1_{E_i}\|_2 \sum_{k=1}^{K} \|\Phi_j 1_{E_k}\|_2\end{aligned}$$

by Lemmas 3 and 1, since each E_i contributes at most $O(N)$ terms of size $p^{-\delta N}$ to $\|K_N(\Phi_j 1_{E_i})\|_{L^2(E_k)}$—were there $x \in E_k$ having CN Hecke points in $\bigcup B_{i,l}$, then there would be have to be $\gtrsim CN$ Hecke points in a single $B_{i,l}$, all at distance $\leqslant N$ in the Hecke tree; this contradicts Lemma 3 once C is large enough. Therefore,

$$\begin{aligned}\big\langle K_N(\Phi_j 1_{\mathcal{E}}), \Phi_j 1_{\mathcal{E}}\big\rangle &\lesssim \sum_{i=1}^{K} \max_{1\leqslant k\leqslant K} Np^{-\delta N}\|\Phi_j 1_{E_i}\|_2 \sum_{k=1}^{K} \|\Phi_j 1_{E_k}\|_2 \\ &\lesssim Np^{-\delta N}\left(\sum_{k=1}^{K}\|\Phi_j 1_{E_k}\|_2\right)^2 \lesssim Np^{-\delta N}\sum_{k=1}^{K}\|\Phi_j 1_{E_k}\|_2^2 \cdot K \lesssim Np^{-\delta N}K \end{aligned} \tag{1}$$

since $\sum_{k=1}^{K}\|\Phi_j 1_{E_k}\|_2^2 \leqslant \|\Phi_j\|_2^2 = 1$.

On the other hand, we can decompose $\Phi_j 1_{\mathcal{E}}$ spectrally into an orthonormal basis (of $L^2(S^*M)$) of T_p eigenfunctions $\{\psi_i\} \ni \Phi_j$, which *a fortiori* also diagonalize K_N, and notice that

$$\Phi_j 1_{\mathcal{E}} = \langle \Phi_j 1_{\mathcal{E}}, \Phi_j\rangle \Phi_j + \sum_{\psi_i \neq \Phi_j} \langle \Phi_j 1_{\mathcal{E}}, \psi_i\rangle \psi_i = \|\Phi_j 1_{\mathcal{E}}\|_2^2 \Phi_j + \sum_{\psi_i \neq \Phi_j} \langle \Phi_j 1_{\mathcal{E}}, \psi_i\rangle \psi_i$$

with

$$\begin{aligned}\sum_{\psi_i \neq \Phi_j} \big|\langle \Phi_j 1_{\mathcal{E}}, \psi_i\rangle\big|^2 &= \|\Phi_j 1_{\mathcal{E}}\|_2^2 - \big|\langle \Phi_j 1_{\mathcal{E}}, \Phi_j\rangle\big|^2 = \|\Phi_j 1_{\mathcal{E}}\|_2^2 - \|\Phi_j 1_{\mathcal{E}}\|_2^4 \\ &< \|\Phi_j 1_{\mathcal{E}}\|_2^2(1-\eta)\end{aligned}$$

by the assumption that $\|\Phi_j 1_{\mathcal{E}}\|_2^2 > \eta$.

Now by Lemma 1, since $\{\psi_i\}$ diagonalizes K_N, and the K_N eigenvalue of each ψ_i is at least -1, we have

$$\begin{aligned}\big\langle K_N(\Phi_j 1_{\mathcal{E}}), \Phi_j 1_{\mathcal{E}}\big\rangle &= \sum_{\psi_i}\big|\langle \Phi_j 1_{\mathcal{E}}, \psi_i\rangle\big|^2 \langle K_N \psi_i, \psi_i\rangle \geqslant \big|\langle \Phi_j 1_{\mathcal{E}}, \Phi_j\rangle\big|^2 \langle K_N \Phi_j, \Phi_j\rangle - \sum_{\psi_i \neq \Phi_j}\big|\langle \Phi_j 1_{\mathcal{E}}, \psi_i\rangle\big|^2 \\ &> \|\Phi_j 1_{\mathcal{E}}\|_2^4 \langle K_N \Phi_j, \Phi_j\rangle - \|\Phi_j 1_{\mathcal{E}}\|_2^2(1-\eta) > \|\Phi_j 1_{\mathcal{E}}\|_2^2\big(\|\Phi_j 1_{\mathcal{E}}\|_2^2 \cdot \eta^{-1} - (1-\eta)\big) \\ &> \eta\big(\eta \cdot \eta^{-1} - 1 + \eta\big) = \eta^2 > 0.\end{aligned} \tag{2}$$

Therefore, combining (1) and (2), we have

$$Np^{-\delta N}K \gtrsim \eta^2$$

and so

$$K \gtrsim \eta^2 N^{-1} p^{\delta N}.$$

Since this holds for any collection of partition elements of total μ-measure $> \eta$, we conclude that there is at most μ-measure η on ergodic components of entropy less than $\delta' \gtrsim \delta > 0$. Taking $\eta \to 0$, we get positive entropy on a.e. ergodic component of μ.

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