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A simple real-variable proof that the Hilbert transform is an L^2-isometry

Une démonstration simple en variables réelles de la propriété d'isométrie L^2 de la transformation de Hilbert H

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ABSTRACT

The Hilbert transform H can be extended to an isometry of L^2. We prove this fact working directly on the principal value integral, completely avoiding the use of the Fourier transform and the use of orthogonal systems of functions. Our approach here is a byproduct of our attempts to understand the rearrangement properties of H.

RÉSUMÉ

La transformation de Hilbert H peut être étendue à une isometrie dans L^2. On demontre cette propriété en utilsant directement la valeur principale de l'intégrale, sans utiliser la transformation de Fourier, ni des systèmes de fonctions orthogonales. L'approche proposée est liée à nos tentative de comprendre le proprietés de réarrangement de H.

1. Introduction

The Hilbert transform H can be defined with the principal value integral,

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x-t)}{t}\,\mathrm{d}t = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|t|>\epsilon} \frac{f(x-t)}{t}\,\mathrm{d}t. \tag{1}$$

The above formula requires some regularity for f, but in fact H can be extended to a linear operator that maps boundedly $L^p(\mathbb{R})$ onto $L^p(\mathbb{R})$ for $1 < p < \infty$. When $p = 2$, it turns out that H is an isometry, namely, for any $f \in L^2(\mathbb{R})$ we have,

$$\|Hf\|_2 = \|f\|_2. \tag{2}$$

This fact follows easily from the representation of H as a Fourier multiplier operator

$$Hf(x) = \int_{-\infty}^{+\infty} (-i\,\mathrm{sgn}\,\xi)\hat{f}(\xi)e^{2\pi i\xi x}\,\mathrm{d}\xi, \tag{3}$$

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but it is not obvious how to obtain it directly from the convolution definition (1) while avoiding (3). In his recent paper [3] J. Duoandikoetxea shows one way to do this, and he points out in the introduction that many "classical" direct proofs of the L^2 boundedness of H as a convolution fail to prove that the best constant is 1, let alone that H is actually an isometry. His technique avoids a direct use of the Fourier transform, but it relies on the orthogonality and completeness of the Hermite functions sequence.

We provide another real-variable proof of (2) that avoids any reference to the Fourier transform and also avoids the use of orthogonal systems of functions. We only use changes of variables in explicit integrals together with approximation in L^2 with step functions. We should mention that our argument here shares some of the same ideas that we used in [2] to give an alternative proof of a theorem of Stein and Weiss: the distribution function of the Hilbert transform of a characteristic function of a set only depends on the Lebesgue measure of such a set.

2. A claim from which the isometry result follows

Let us denote by $\chi_{[a,b]}$ the characteristic function of an interval. Clearly we have

$$H\chi_{[a,b]}(x) = \frac{1}{\pi}\int_a^b \frac{\mathrm{d}t}{x-t} = \frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right|.$$

We *claim* that the isometry property (2) holds in two very special cases: when f is the characteristic function of one interval, and when f is the characteristic function of the union of two disjoint intervals. Namely, assuming $a < b < c < d$, we claim that

$$\|H\chi_{[a,b]}\|_2^2 = \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\left(\log\left|\frac{x-a}{x-b}\right|\right)^2 \mathrm{d}x = b-a; \tag{4}$$

$$\|H\chi_{[a,b]\cup[c,d]}\|_2^2 = \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\left(\log\left|\frac{x-a}{x-b}\right| + \log\left|\frac{x-c}{x-d}\right|\right)^2 \mathrm{d}x = (b-a)+(d-c). \tag{5}$$

Assuming the above let us proceed towards our main goal. First we observe that (4) and (5) imply the following orthogonality property for the Hilbert transforms of the characteristic functions of two disjoint intervals

$$\begin{aligned} 2\int_{-\infty}^{+\infty} H\chi_{[a,b]}(x)\cdot H\chi_{[c,d]}(x)\,\mathrm{d}x &= \int_{-\infty}^{+\infty}\left(H\chi_{[a,b]}(x) + H\chi_{[c,d]}(x)\right)^2 \mathrm{d}x \\ &\quad - \int_{-\infty}^{+\infty}\left(H\chi_{[a,b]}(x)\right)^2 \mathrm{d}x - \int_{-\infty}^{+\infty}\left(H\chi_{[c,d]}(x)\right)^2 \mathrm{d}x = 0. \end{aligned} \tag{6}$$

We then consider a general simple step function $g(x) = \sum_{k=1}^m \alpha_k \chi_{[a_k,b_k]}(x)$ where the m intervals $[a_k, b_k]$ are essentially disjoint, and the coefficients α_k are real. We obtain,

$$\|Hg\|_2^2 = \int_{-\infty}^{+\infty}\left(\sum_{k=1}^m \alpha_k H\chi_{[a_k,b_k]}(x)\right)^2 \mathrm{d}x = \sum_{k=1}^m \alpha_k^2 (b_k - a_k) = \|g\|_2^2, \tag{7}$$

because expanding the square in the second term of the above equality we get that the "mixed" terms are zero by (6), while each square term contributes one positive term to the sum on the r.h.s. by (4).

Finally, it is well known that any $f \in L^2(\mathbb{R})$ can be approximated arbitrarily well by a step function g chosen as in (7) and it follows that (2) holds in general.

3. Proof of the claim

Formula (4) can be written:

$$\|H\chi_{[a,b]}\|_2^2 = \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\left(\log\left|\frac{x-a}{x-b}\right|\right)^2 \mathrm{d}x = \frac{1}{\pi^2}\int_{-\infty}^{+\infty}\left(\log\left|\frac{x-b}{x-a}\right|\right)^2 \mathrm{d}x.$$

The function $y = \frac{x-b}{x-a}$ is monotonic increasing for all $x \neq a$ and intersects the line $y = t$ in the point $x(t) = \frac{at-b}{t-1}$. A change of variable therefore yields:

$$\|H\chi_{[a,b]}\|_2^2 = \frac{1}{\pi^2}\int_{-\infty}^{+\infty} \big(\log|t|\big)^2 x'(t)\,\mathrm{d}t = \frac{1}{\pi^2}\int_{-\infty}^{+\infty} \frac{(\log|t|)^2}{(t-1)^2}\,\mathrm{d}t\cdot(b-a). \tag{8}$$

Formula (5) can be written

$$\|H\chi_{[a,b]\cup[c,d]}\|_2^2 = \frac{1}{\pi^2}\int_{-\infty}^{+\infty} \left(\log\left|\frac{(x-b)(x-d)}{(x-a)(x-c)}\right|\right)^2 \mathrm{d}x. \tag{9}$$

The function $y = \frac{(x-b)(x-d)}{(x-a)(x-c)}$ is monotonic increasing for all $x \neq a, c$ and intersects the line $y = t$ in two points $x_1(t)$ and $x_2(t)$ which are the two solutions of the second degree equation $(x-b)(x-d)t = (x-a)(x-c)$ with parameter t. We can label them in such a way that $x_1(t)$ provides a bijection between $(-\infty, +\infty)$ and (a, c) while $x_2(t)$ provides a bijection between $(-\infty, +\infty)$ and $(-\infty, a) \cup (c, +\infty)$. From the equation it is easy to see that $x_1(t) + x_2(t) = \frac{(b-a)+(d-c)}{(1-t)}$ and therefore $x_1'(t) + x_2'(t) = \frac{(b-a)+(d-c)}{(1-t)^2}$. We have:

$$\begin{aligned}
\|H\chi_{[a,b]\cup[c,d]}\|_2^2 &= \frac{1}{\pi^2}\left[\int_{(a,c)} \left(\log\left|\frac{(x-b)(x-d)}{(x-a)(x-c)}\right|\right)^2 \mathrm{d}x + \int_{(-\infty,a)\cup(c,+\infty)} \left(\log\left|\frac{(x-b)(x-d)}{(x-a)(x-c)}\right|\right)^2 \mathrm{d}x\right] \\
&= \frac{1}{\pi^2}\int_{-\infty}^{+\infty} \big(\log|t|\big)^2\big[x_1'(t) + x_2'(t)\big]\,\mathrm{d}t = \frac{1}{\pi^2}\int_{-\infty}^{+\infty} \frac{(\log|t|)^2}{(t-1)^2}\,\mathrm{d}t\cdot\big[(b-a)+(d-c)\big].
\end{aligned} \tag{10}$$

The integral factor $\frac{1}{\pi^2}\int_{-\infty}^{+\infty} \frac{(\log|t|)^2}{(t-1)^2}\,\mathrm{d}t$ appears both in (8) and (10). If we show that this factor is equal to 1, then our claim follows. We have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{(\log|t|)^2}{(t-1)^2}\,\mathrm{d}t &= \int_0^{+\infty} (\log t)^2\left[\frac{1}{(t-1)^2} + \frac{1}{(t+1)^2}\right]\mathrm{d}t \\
&= \int_0^1 (\log t)^2\left[\frac{1}{(t-1)^2} + \frac{1}{(t+1)^2}\right]\mathrm{d}t + \int_0^1 \left(\log\frac{1}{u}\right)^2\left[\frac{1}{(\frac{1}{u}-1)^2} + \frac{1}{(\frac{1}{u}+1)^2}\right]\frac{1}{u^2}\,\mathrm{d}u \\
&= 2\int_0^1 (\log t)^2\left[\sum_{k=0}^{+\infty}(k+1)t^k + \sum_{k=0}^{+\infty}(-1)^k(k+1)t^k\right]\mathrm{d}t \\
&= 4\int_0^1 (\log t)^2\sum_{n=0}^{+\infty}(2n+1)t^{2n}\,\mathrm{d}t = 4\int_0^{+\infty}(-s)^2\sum_{n=0}^{+\infty}(2n+1)e^{-(2n+1)s}\,\mathrm{d}s \\
&= 8\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = 8\frac{\pi^2}{8} = \pi^2.
\end{aligned}$$

Our claim holds, together with the main result.

4. Final remarks

We can check, avoiding the use of the Fourier transform, also that $H : L^2 \to L^2$ is a surjective map. One way to do this is to show, starting from (1) that H is an anti-self adjoint operator, namely $\langle Hf, g\rangle = -\langle f, Hg\rangle$ where $\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)\,\mathrm{d}x$. The above identity holds when f and g are Schwartz functions and it can be extended to L^2 in a standard way. If (2) were just a partial isometry, there would be a function $g_0 \in L^2$ orthogonal to Hf for all $f \in L^2$. In other words $\langle Hf, g_0\rangle = -\langle f, Hg_0\rangle = 0$, but this implies $g_0 \equiv 0$. We have given before a self-contained proof of (4) and (5), but actually they are both special cases of Lemma 2.2 in [2], where we show that for any $1 < p < \infty$, we have:

$$\int_{-\infty}^{+\infty} \left|H\left(\sum_{k=1}^{n} \chi_{[a_k,b_k]}(x)\right)\right|^p \mathrm{d}x = \frac{4(1-2^{-p})}{\pi^p}\zeta(p)\Gamma(p+1)\sum_{k=1}^{n}(b_k - a_k).$$

This formula expresses a rearrangement property of the Hilbert transform, because if we apply H to a step function f whose steps are all at the same height then the norm $\|Hf\|_p$ is equal to the total length of its steps times some explicit function of p and this is invariant with respect to the horizontal positions of these steps. Also (7) expresses a rearrangement property, telling us that if we apply the Hilbert transform H to a step function f whose steps have *arbitrary* height then the L^2 norm $\|Hf\|_2$ is equal to $\|f\|_2$ and, in particular, it is invariant under re-shuffling of the horizontal positions of its steps (at different heights) provided we avoid overlapping.

The norm $\|Hf\|_p$ for f a step function f whose steps have *arbitrary* height, is maximized or minimized in correspondence to certain symmetric rearrangements of its steps, and the situation is different when $p > 2$ or $p < 2$. We conjecture that the configurations that maximize the L^p norm when $p > 2$, minimize it when $p < 2$, and vice versa. The case $p = 2$ is very special with respect to these properties, because the minimizers coincide with the maximizers. We omit our precise putative statements here because this is still work in progress, but we point out that a rearrangement theorem that goes in this direction (for the case $p < 2$ and for the conjugate operator on the circle instead of H on the line) can be found in [1]. Another related result is our Corollary 2, p. 319, in [4].

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