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Numerical Analysis

Aitken's acceleration of the Restricted Additive Schwarz preconditioning using coarse approximations on the interface

Accélération de Aitken du préconditionnement Schwarz Additif Restreint utilisant des approximations grossières de l'interface

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A B S T R A C T

An enhancement of the restricted Additive Schwarz (RAS) preconditioning, based on the Aitken's acceleration of the convergence of the Schwarz method, is proposed. Its numerical performance is compared with the RAS preconditioning on the two dimensional Helmholtz problem.

R É S U M É

Une amélioration du préconditionneur Schwarz Additif Restreint (RAS) fondée sur l'accélération de la convergence purement linéaire de la méthode de Schwarz par la méthode de Aitken, est proposée. Sa performance est comparée au préconditionneur RAS sur le problème de Helmholtz bidimensionnel.

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1. Introduction

The convergence rate of a Krylov method such as the Generalized Conjugate Residual (GCR) [4], to solve a linear system $Au = f$, $A = (a_{ij}) \in \mathbb{R}^{m\times m}$, $u \in \mathbb{R}^m$, $b \in \mathbb{R}^m$, decreases with increasing condition number $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$ of the non-singular matrix A. Left preconditioning techniques consist to solve $M^{-1}Au = M^{-1}f$ such that $\kappa_2(M^{-1}A) \ll \kappa_2(A)$. The Additive Schwarz (AS) preconditioning is built from the adjacency graph $G = (W, E)$ of A, where $W = \{1, 2, \ldots, m\}$ and $E = \{(i, j)\colon a_{ij} \neq 0\}$ are the edges and vertices of G. Starting with a non-overlapping partition $W = \bigcup_{i=1}^{p} W_{i,0}$ and $\delta \geqslant 0$ given, the overlapping partition $\{W_{i,\delta}\}$ is obtained defining p partitions $W_{i,\delta} \supset W_{i,\delta-1}$ by including all the immediate neighbouring vertices of the vertices in the partition $W_{i,\delta-1}$. Then the restriction operator $R_{i,\delta} : W \to W_{i,\delta}$ defines the local operator $A_{i,\delta} = R_{i,\delta} A R_{i,\delta}^T$, $A_{i,\delta} \in \mathbb{R}^{m_{i,\delta}\times m_{i,\delta}}$ on $W_{i,\delta}$. The AS preconditioning writes: $M_{AS,\delta}^{-1} = \sum_{i=1}^{p} R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta}$. Introducing $\tilde{R}_{i,\delta}$ the restriction matrix on a non-overlapping subdomain $W_{i,0}$, the Restricted Additive Schwarz (RAS) iterative process [1] writes:

$$u^k = u^{k-1} + M_{RAS,\delta}^{-1}\bigl(f - Au^{k-1}\bigr), \quad \text{with } M_{RAS,\delta}^{-1} = \sum_{i=1}^{p} \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} \tag{1}$$

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The RAS exhibits a faster convergence than the AS, as shown in [3], leading to a better preconditioning that depends of the number of subdomains. When it is applied to linear problems, the RAS has a pure linear rate of convergence/divergence that can be enhanced with optimized boundary conditions giving the ORAS method of [2]. The RAS method's linear convergence allows its acceleration of the convergence by the Aitken's process as done in [6] for the Schwarz method.

In what follows, we write the Aitken Restricted Additive Schwarz (ARAS) iterative process and the associated preconditioner. This preconditioner belongs to the family of the two level preconditioner techniques (see [8,9] and references) but the coarse grid operator uses only parts of the artificial interfaces contrary to the patch substructuring method of [5]. In this way, it can be seen as similar as the SchurRAS method of [7] but it differs because the discrete Steklov–Poincaré operator connects the coarse artificial interfaces of all the subdomains. Numerical results of the good properties of the ARAS preconditioning are provided.

2. Aitken's acceleration of RAS preconditioning

Let $\Gamma_i = (I_{m_{i,\delta}} - R_{i,\delta}^T)W_{i,\delta}$ be the interface associated to $W_{i,\delta}$ and $\Gamma = \bigcup_{i=1}^p \Gamma_i$ be the global interface. Then $u_{|\Gamma} \in \mathbb{R}^n$ is the restriction of the solution $u \in \mathbb{R}^m$ on the Γ interface and $e_{|\Gamma}^k = u_{|\Gamma}^k - u_{|\Gamma}^\infty$ is the error of (1) at the interface Γ. The pure linear behaviour of the error on the interface Γ of the RAS method conducts to the existence of a matrix $P \in \mathbb{R}^{n\times n}$ independent of the iterate k such as $e_{|\Gamma}^k = Pe_{|\Gamma}^{k-1}$. Then, we can apply the Aitken's acceleration of the convergence process [6] (if $\|P\| < 1$ to ensure existence of $(I_n - P)^{-1}$ for example) as follows:

$$u_{|\Gamma}^\infty = (I_n - P)^{-1}\big(u_{|\Gamma}^k - Pu_{|\Gamma}^{k-1}\big) \tag{2}$$

P can be computed analytically or numerically for a separable operator on separable geometry [6] or numerically approximated in other cases [10]. Using this property on the RAS method, we would like to write a preconditioner which includes the Aitken's acceleration process. We introduce a restriction operator $R_\Gamma \in \mathbb{R}^{n\times m}$ from W to the global artificial interface Γ, with $R_\Gamma R_\Gamma^T = I_n$. The Aitken Restricted Additive Schwarz (ARAS) must generate a sequence of solution on the interface Γ, and accelerate the convergence of the Schwarz process from this original sequence. Then the accelerated solution on the interface replaces the last one. This could be written combining an AS or RAS process Eq. (3a) with the Aitken process written in $\mathbb{R}^{m\times m}$ Eq. (3b) and subtracting the Schwarz solution which is not extrapolated on Γ Eq. (3c). We can write the following approximation u^* of the solution u:

$$u^* = u^{k-1} + M_{RAS,\delta}^{-1}\big(f - Au^{k-1}\big) \tag{3a}$$
$$+ R_\Gamma^T(I_n - P)^{-1}\big(u_{|\Gamma}^k - Pu_{|\Gamma}^{k-1}\big)R_\Gamma \tag{3b}$$
$$- R_\Gamma^T I_n R_\Gamma\big(u^{k-1} + M_{RAS,\delta}^{-1}\big(f - Au^{k-1}\big)\big) \tag{3c}$$

This formulation gives the ARAS iterative process:

$$u^k = u^{k-1} + \big(I_m + R_\Gamma^T\big((I_n - P)^{-1} - I_n\big)R_\Gamma\big)M_{RAS,\delta}^{-1}\big(f - Au^{k-1}\big) \tag{4}$$

Then we defined the ARAS preconditioner as

$$M_{ARAS,\delta}^{-1} = \big(I_m + R_\Gamma^T\big((I_n - P)^{-1} - I_n\big)R_\Gamma\big)\sum_{i=1}^p \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} \tag{5}$$

If P is known exactly, the ARAS process written in Eq. (4) needs two steps to converge to the solution with an initial guess $u^0 = 0$. Then we have:

Proposition 2.1. *If P is known exactly then we have $A^{-1} = (2M_{ARAS,\delta}^{-1} - M_{ARAS,\delta}^{-1}AM_{ARAS,\delta}^{-1})$ that leads $(I - M_{ARAS,\delta}^{-1}A)$ to be a nilpotent matrix of degree 2.*

Hence, if P is known exactly there is no need to use ARAS as a preconditioning technique. Nevertheless, when P is approximated, the Aitken's acceleration of the convergence depends on the local domain solving accuracy, and the cost of the building of an exact P depends on the size n. This is why P is numerically approximated by $P_{\mathbb{U}_q}$ as in [10], defining $q \leqslant n$ orthogonal vectors $\mathbb{U}_q \in R^{n\times q}$ distributed uniformly on each artificial interface. Then it makes sense to use it as a preconditioning technique. We set a reduction factor $r = \frac{n}{q}$ which defines the coarse interface size and build different preconditioners ARAS(r),

$$M_{ARAS(r),\delta}^{-1} = \big(I_m + R_\Gamma^T\mathbb{U}_q\big((I_q - P_{\mathbb{U}_q})^{-1} - I_q\big)\mathbb{U}_q^T R_\Gamma\big)\sum_{i=1}^p \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} \tag{6}$$

From Proposition 2.1 we build a better preconditioner denoted by ARAS2:

$$M_{ARAS2(r),\delta}^{-1} = 2M_{ARAS(r),\delta}^{-1} - M_{ARAS(r),\delta}^{-1}AM_{ARAS(r),\delta}^{-1} \tag{7}$$

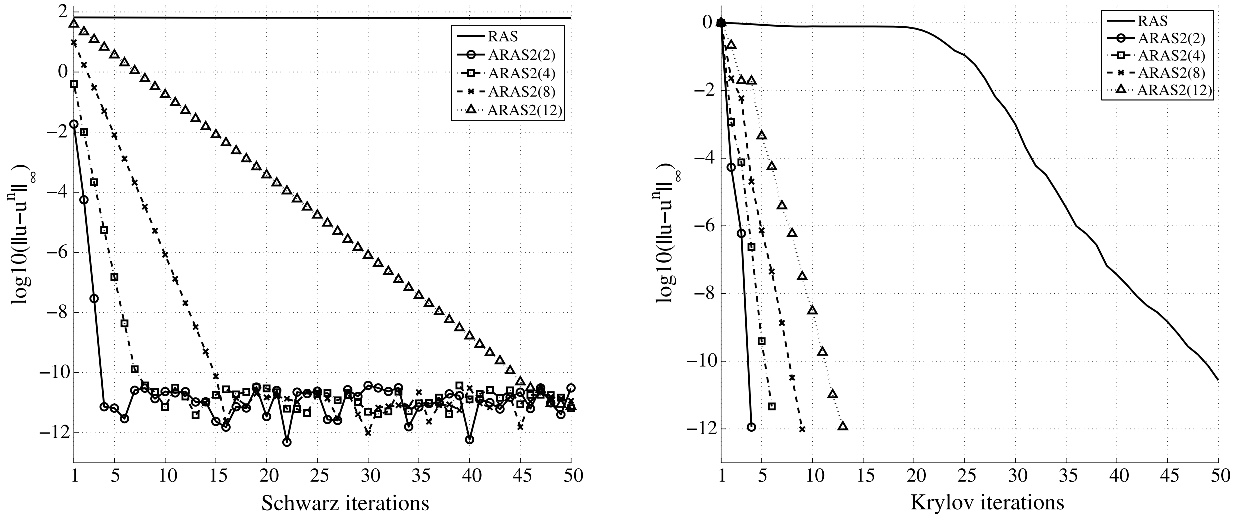

Fig. 1. Solving 2D Helmholtz equation on a 164×164 Cartesian grid, $p = 8$, $r = \{2, 4, 8, 12\}$, (left) convergence of Iterative (Aitken)-RAS2 processes to the solution, (right) GCR preconditioned by (Aitken)-RAS2.

Fig. 1. Résolution de l'équation de Helmholtz 2D sur une grille 164×164, $p = 8$, $r = \{2, 4, 8, 12\}$, (gauche) convergence des processus iteratifs (Aitken)-RAS2 vers la solution, (droite) RCG préconditionné par (Aitken)-RAS2.

Table 1
Estimation using the lapack routine DGECON of $\kappa_\infty(M^{-1}_{ARAS2(r)}A)$ and $\kappa_\infty(M^{-1}_{RAS}A)$ for different problem sizes m and number of subdomains p and $r = \{1, 2, 4, 8, 12\}$.

Tableau 1
Estimation de $\kappa_\infty(M^{-1}_{ARAS2(r)}A)$ et $\kappa_\infty(M^{-1}_{RAS}A)$, en utilisant la routine lapack DGECON pour différentes tailles du problème m et nombres de sous domaines p et $r = \{1, 2, 4, 8, 12\}$.

m	p	$\kappa_\infty(M^{-1}_{RAS}A)$	$\kappa_\infty(M^{-1}_{ARAS2(1)}A)$	$\kappa_\infty(M^{-1}_{ARAS2(2)}A)$	$\kappa_\infty(M^{-1}_{ARAS2(4)}A)$	$\kappa_\infty(M^{-1}_{ARAS2(8)}A)$	$\kappa_\infty(M^{-1}_{ARAS2(12)}A)$
64	2	570.7825	1.0000	1.0967	1.4245	4.4708	11.9134
64	4	1.4813 E+03	1.0000	1.6427	4.4445	59.5047	192.4787
164	4	3.7253 E+03	1.0000	1.5282	3.5500	34.7914	158.4591
164	**8**	*7.3038* **E+03**	**1.0000**	**4.3868**	**25.6287**	**499.7077**	1.3970 **E+03**
164	16	1.4532 E+04	1.0000	35.1335	360.3884	3.5580 E+03	1.5716 E+03

3. Numerical results on an ill-conditioned Helmholtz problem

Let us consider the 2D Helmholtz problem $(-\omega - \triangle)u = f$ in $\Omega = [0, 1]^2$, $u = 0$ on $\partial\Omega$. The problem is discretized by second order finite differences with m points in each direction x and y giving a space step $h = \frac{1}{m-1}$. The set value $\omega = 0.98\frac{4}{h^2}(1 - \cos(\pi h))$ is close to the minimum eigenvalue of the discrete $-\triangle$ operator in order to have an ill-conditioned discrete problem with $\kappa_\infty(A) = 1.7918\,\text{E}{+}07$ for $m = 164$. Fig. 1 shows the convergence of the RAS and ARAS2(r) iterative processes for $r = \{2, 4, 8, 12\}$ and $m = 164$ and their application as preconditioners to the GCR method for a decomposition in the y direction into 8 subdomains with an overlap equal to $3h$. Enhancement of the Krylov method preconditioned by ARAS2 is effective, even for a coarse interface of size $\frac{n}{12}$. For $r = \{2, 4\}$ the Krylov method does not change the extremely fast convergence of the Aitken process while the Krylov method becomes effective for $r > 4$. Table 1 gives the condition number of the preconditioned system with RAS and ARAS2(r) for different number of subdomains $p = \{2, 4, 8, 16\}$ and $m = \{64, 164\}$. $\kappa_\infty(M^{-1}_{ARAS2(1)}A)$ provides the optimal preconditioning (because ARAS2 ($r = 1$) converges in one step as $q = n$, $P_{\mathbb{U}_n} = P$ and therefore Proposition 2.1 applies).

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