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Algebra

A characterization of generalized quaternion 2-groups

Une caractérisation des groupes de quaternions généralisés

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ABSTRACT

The goal of this Note is to give a characterization of generalized quaternion 2-groups by using their posets of cyclic subgroups.

RÉSUMÉ

Le but de cette Note est de donner une caractérisation des 2-groupes de quaternions généralisés en utilisant leur ensembles partiellement ordonnés de sous-groupes cycliques.

1. Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G. The starting point for our discussion is given by the paper [1], where the proper nontrivial subgroups H of G with the property that

for every $X \in L(G)$ we have $X \leqslant H$ or $H \leqslant X$

have been studied. Such a subgroup is called a *breaking point* for the lattice $L(G)$. Clearly, if $L(G)$ is a chain (i.e. G is a cyclic p-group), then all proper nontrivial subgroups H of G are breaking points. On the other hand, we remark that the above concept can naturally be extended to other remarkable posets of subgroups of G (and also to arbitrary posets). One of them is the poset of cyclic subgroups of G, denoted usually by $C(G)$. The study of the existence and of the uniqueness of breaking points in $C(G)$ constitutes the purpose of this paper.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on group theory can be found in [2] and [4]. For subgroup lattice notions we refer the reader to [3] and [5].

We mention that by a *generalized quaternion* 2*-group* we mean a group of order 2^n for some natural number $n \geqslant 3$, defined by the presentation

$$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2,\ a^{2^{n-1}} = 1,\ b^{-1}ab = a^{-1} \rangle.$$

We also recall that these groups are the unique finite noncyclic p-groups all of whose abelian subgroups are cyclic, or equivalently the unique finite noncyclic p-groups possessing exactly one subgroup of order p (see (4.4) of [4], II). Obviously, this result shows that the subgroup of order 2 of Q_{2^n}, namely $\langle a^{2^{n-2}} \rangle$, is the unique breaking point of $C(Q_{2^n})$.

Our main theorem proves that generalized quaternion 2-groups exhaust all finite noncyclic groups whose posets of cyclic subgroups have breaking points.

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Theorem 1.1. *Let G be a finite group. Then $C(G)$ possesses breaking points if and only if G is either a cyclic p-group of order at least p^2 or a generalized quaternion 2-group.*

2. The proof of Theorem 1.1

We observe first that the above theorem can be easily proved in the particular case of p-groups.

Lemma 2.1. *Let G be a finite p-group. Then $C(G)$ possesses breaking points if and only if G is either a cyclic p-group of order at least p^2 or a generalized quaternion 2-group.*

Proof. Suppose that G is not cyclic and let H be a breaking point of $C(G)$. Then all minimal subgroups $M_1, M_2, \ldots, M_k$ of G are contained in H. If $k \geqslant 2$, then we infer that H is not cyclic, a contradiction. So, we have $k = 1$, that is G has a unique subgroup of order p. This implies that G is a generalized quaternion 2-group, according to the result mentioned in Section 1.

The converse implication is obvious, completing the proof. □

We are now able to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that the poset $C(G)$ of cyclic subgroups of a finite group G possesses a breaking point, say H.

In the following we shall focus on proving that G must necessarily be a p-group. By the way of contradiction, assume that the order of G has at least two distinct prime divisors. Clearly, the same thing can be also said about the order of H. Let $p \in \pi(G)$ and K be a cyclic p-subgroup of G. Since H is not a p-subgroup, we infer that $K \subseteq H$. In other words, H contains any cyclic p-subgroup of G and consequently any p-element of G. This implies that all Sylow p-subgroups of G are contained in H. Then $H = G$, a contradiction.

Hence G is a p-group, for some prime p, and now the conclusion follows from Lemma 2.1. □

By the above results we also infer that, given a finite group G, the poset $C(G)$ possesses a *unique* breaking point if and only if G is either a cyclic p-group of order p^2 or a generalized quaternion 2-group. In other words, the following corollary holds.

Corollary 2.2. *The generalized quaternion 2-groups are the unique finite noncyclic groups whose posets of cyclic subgroups have exactly one breaking point.*

Finally, we indicate a natural generalization of our study, suggested by the reviewers of the paper. Let G be a finite group and denote by

$$\overline{C}(G) = \left\{[H] \,\middle|\, H \in C(G)\right\}$$

the set of conjugacy classes of cyclic subgroups of G. Mention that $\overline{C}(G)$ is also a poset under the ordering relation

$$[H_1] \leqslant [H_2] \quad \text{if and only if} \quad H_1 \subseteq H_2^g, \quad \text{for some } g \in G.$$

Take a breaking point $[H]$ of $\overline{C}(G)$. Then $H \in C(G)$ satisfies the following condition: for any cyclic subgroup C of G, some conjugate of C in G contains or is contained in H. Clearly, this is weaker than the condition that H be a breaking point of $C(G)$. We remark that for a finite p-group G it is sufficient to guarantee the uniqueness of a subgroup of order p in G. In other words, Lemma 2.1 also holds if we replace $C(G)$ with $\overline{C}(G)$. In the general case, that is for *arbitrary* finite groups G, the problem of characterizing the existence and the uniqueness of breaking points of $\overline{C}(G)$ remains still open.

3. Conclusions and further research

All previous results show that the concept of breaking point in some posets of subgroups of a (finite) group G can constitute an important aspect of subgroup lattice theory. Clearly, its study started in [1] for lattices of subgroups and continued in the present paper for posets of cyclic subgroups can successfully be extended to other significant lattices/posets associated to G (as the lattice of normal/subnormal/characteristic/solitary subgroups of G or the poset of centralizers/conjugacy classes of elements (subgroups) of G). Studying the breaking points of arbitrary posets (not necessarily connected with a group G) seems to be also very interesting. These will surely constitute the subject of some further research.

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