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Dynamical Systems

Topological equivalence of vector fields after blow-up

Équivalence topologique de champs de vecteurs après éclatement

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A B S T R A C T

We give a theorem of characterization for the property of being *topologically equivalent after blow-up* in the set of germs of three-dimensional hyperbolic vector fields. Given ξ, ξ' two such a germs and Φ a finite sequence of blow-ups of the ambient space, we find, under non-resonance conditions associated to Φ, a criterion that permits to determine if there is a topological equivalence between ξ and ξ' that lifts to Φ. We deduce that there are only finitely many possible classes of Φ-topological equivalence in the considered set of vector fields.

R É S U M É

Nous donnons un théorème de caractérisation pour la propriété d'être *topologiquement équivalents après éclatement* dans l'ensemble des germes de champ de vecteurs hyperboliques en dimension trois. Etant donnés deux tels germes ξ, ξ', et Φ une suite d'éclatements de l'espace ambiant, on trouve, sous des conditions de non-résonance associées à Φ, un critère permettant de déterminer s'il existe une équivalence topologique entre ξ et ξ' se relevant à Φ. Nous en déduisons qu'il n'y a qu'un nombre fini de classes de Φ-équivalence topologique dans l'ensemble des champs de vecteurs considéré.

Let us denote by ξ and ξ' two germs of analytic vector fields at the origin of $\mathbb{R}^3$. We assume that the origin is an equilibrium point (a singularity) both for ξ and ξ'. Recall that ξ and ξ' are $\mathcal{C}^r$-equivalent if there exists a $\mathcal{C}^r$-diffeomorphism of $(\mathbb{R}^3, 0)$ that sends trajectories of ξ into trajectories of ξ' preserving the orientation.

This note concerns a concept of topological equivalence that lifts to a finite sequence of blow-ups of the ambient space. It is a geometrical definition intermediate between topological equivalence and differentiable equivalence. Note that, already in dimension two, there are topological equivalences between $\xi = x\partial x + 2y\partial y$ and $\xi' = x\partial x + 3y\partial y$ that lift after one blow-up, but none of them lifts after two blow-ups. On the other hand, the $\mathcal{C}^1$-equivalence of diagonal linear vector fields implies that they have the same linear part up to constant factor.

Consider an application

$$\Phi : \tilde{M} = M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_1} M_0 = \left(\mathbb{R}^3, 0\right)$$

where π_j is the blow-up with center $P_{j-1} \in M_{j-1}$. We say that ξ and ξ' are *Φ-topologically equivalent* if:

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(i) Each P_j is a singular point for the respective transformed ξ_j, ξ'_j of ξ_{j-1}, ξ'_{j-1} by π_j. Following the terminology used in [4,5], we say that Φ is a *sequence of infinitely near singular points for* ξ *and* ξ'.
(ii) There are representatives of ξ, ξ' defined in respective neighborhoods V, V' and a topological equivalence $h: V \to V'$ between ξ and ξ' that lifts to a homeomorphism $\tilde{h}: \Phi^{-1}(V) \to \Phi^{-1}(V')$. Note that $\tilde{h}$ gives automatically a topological equivalence between ξ_n and ξ'_n.

In this note, we consider *hyperbolic germs of vector field*, that is, all the eigenvalues of the linear parts have non-zero real part. We also take finitely many non-resonance conditions on the eigenvalues depending on the fixed sequence Φ. In particular, the eigenvalues will have different real parts, hence they are all real and the linear parts can be made diagonal. Moreover, we shall assume that the three coordinate planes are invariant. This is not a so restrictive condition. For instance, in the case of an attractor (or a repeller), by Poincaré's Theorem [6, pp. xcix-cv] the vector field is analytically equivalent to its linear part (except in presence of resonances) and hence we have three invariant planes (assuming it is linear); also in the case of a saddle, we can apply the same argument to the invariant variety of dimension two and, following small transversal curves to it by the flow, we can get two more (topological) invariant planes.

Let us give a precise definition of the class of vector fields we work with. Denote $H = H_1 \cup H_2 \cup H_3$ the coordinate planes of $M_0 = \mathbb{R}^3$. We are going to give an inductive definition of a (Φ, H)*-vector field*. We say that ξ is an H*-vector field* iff the planes H_i are invariant and the three eigenvalues are distinct and non-zero. If $n = 0$ a (Φ, H)-vector field is by definition an H-vector field. Note that the blow-up ξ_1 of an H-vector field ξ has exactly three singular points $Q_1, Q_2, Q_3 \in \pi_1^{-1}(0)$, corresponding to the strict transform of the coordinate lines and also $\pi_1^{-1}(0)$ is invariant by ξ_1. Hence we obtain a new set of three invariant planes $H'^{(j)}$ at each Q_j, for $j = 1, 2, 3$. If $n > 0$, we say that ξ is a (Φ, H)*-vector field* iff ξ is an H-vector field and ξ_1 is a $(\Phi_j, H'^{(j)})$-vector field at Q_j, for $j = 1, 2, 3$, where Φ_j is the germ of $\pi_2 \circ \cdots \circ \pi_n$ at Q_j.

Let us note that any H-vector field ξ without integer resonances in the eigenvalues is a (Φ, H)-vector field for any sequence Φ of infinitely near singular points. Indeed, given a fixed Φ obtained by blowing-up corners in H, there is a finite number of equations of integer resonances such that an H-vector field ξ is a (Φ, H)-vector field if and only if its linear part avoids these resonances.

In this note we give an idea of the proof of the following theorem:

Theorem 1. *There are only finitely many classes of Φ-topological equivalence in the set of (Φ, H)-vector fields. Moreover, the Φ-equivalence class of a (Φ, H)-vector field is given by its linear part.*

We take the viewpoint of varieties with corners without making explicit mention of it. For instance, our initial ambient space is $(\mathbb{R}_{\geqslant 0}, 0)$ and H is its boundary. The exceptional divisor $D = \Phi^{-1}(0)$ jointly with the transform $\tilde{H}$ of H is the boundary of M_n. Recall that $\Phi: \tilde{M} \setminus D \to \mathbb{R}^3 \setminus \{0\}$ is an analytic isomorphism. As a consequence of the definition of a (Φ, H)-vector field, all the singularities of ξ_n are isolated, hyperbolic and corners of $D \cup \tilde{H}$. Moreover, if the starting singularity (the origin) is a saddle, the singular points of ξ_n on D are all saddles. However, if we start with an attractor (or a repeller) at the origin, there is a unique attractor $\mathcal{Q}(\xi, \Phi)$ (respectively a repeller) and the remaining ones are saddles.

Dynamics over the exceptional divisor

The dynamics of the restriction of ξ_n to a component D_i of D is always the same: there is an attractor A_i, one repeller R_i, and at least one saddle point. However, the dynamics of ξ_n on $\tilde{H}_i$ is just the blow-up of the dynamics on H_i. Each edge $D_i \cap D_j$ of D has an orientation induced by ξ_n. It is important to remark that we do not produce cycles of oriented edges.

Weights transition

Suppose that the linear part of an H-vector field ξ whit a saddle point at Q is

$$L\xi = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z}$$

whit $\lambda\mu\delta < 0$ and $\mu\delta > 0$ in $x, y, z \geqslant 0$. The *intrinsic* (y, z)*-weight* of Q is δ/μ (see [1]) and we say that it is attached to the x-axis. Now we describe a process of weights transition $\rho \to \rho'$ introduced in [2,3]. Consider the curve $x = 1, z = y^{\frac{\delta}{\mu}}$. The saturation of this curve by the flow accumulates at the "middle" of the invariant variety $x = 0$. Take now a curve $x = 1$, $z = y^{\rho}$ with $\rho > \delta/\mu$ representing a weight ρ attached to the x-axis, its saturation accumulates at the y-axis and contains the curve $y = 1$, $z = x^{\rho'}$, where $\rho' = (\delta - \mu\rho)/\lambda$. We say that ρ *transits* to ρ'. The situation is similar if we have $\rho < \delta/\mu$. If $\rho = \delta/\mu$ we say that there is *no transition*. Doing also the inverse process we get all the *weights transitions* through the saddle Q.

(Φ, ξ)-weights and saddle-connections

Consider a divisor component D_i and take the two-dimensional attractor A_i of D_i. We start with the intrinsic weight α_i of A_i. According to the previous transition rules, the weight α_i transits by the flow through a finite number of saddles producing an associated weight at each step (in our configuration there are not return situations). Two possibilities can arrive:

(i) The process does not stop at any saddle. It finishes at one of the three transformed of the initial coordinate axes or at the point $\mathcal{Q}(\xi, \Phi)$ if it is an attractor. We define β_i as the output weight from α_i. These weights β_i form the set $\mathcal{W}(\Phi, \xi)$ of the (Φ, ξ)-*weights*. Note that an element $\beta_i \in \mathcal{W}(\Phi, \xi)$ contains the information of the original D_i as well as the information of the final edge to which it is attached. If we find a two-dimensional attractor in one of the $\tilde{H}_i$, we add a weight as above.
(ii) There is a saddle where the process stops. This saddle has to be necessarily the repeller R_j of a component D_j (the arriving value from A_i coincides with the intrinsic weight of R_j). In this case, we say that there is a *saddle-connection* $S(D_i, D_j)$ (see [1]). Recall that, as divisor components, the transformed of H could also take part in a saddle-connection.

Notice that if there is a saddle-connection $S(D_i, D_j)$ we do not associate any weight to the component D_i. There can appear consecutive saddle-connections although the absence of cycles on D implies that two divisor components are never doubly connected.

Lemma 2. *The saddle-connections, the position and orientation of $\mathcal{Q}(\xi, \Phi)$ and the set $\mathcal{W}(\Phi, \xi)$ depend only on the eigenvalues of the linear part of ξ at the origin.*

We could also define a set of weights from the repellers R_i following the flow in negative sense. This data would give to us the same kind of information as $\mathcal{W}(\Phi, \xi)$.

Given two (Φ, H)-vector fields ξ and ξ', we say that the sets $\mathcal{W}(\Phi, \xi)$ and $\mathcal{W}(\Phi, \xi')$ are *similar* when the following conditions hold: if $\beta_i \in \mathcal{W}(\Phi, \xi)$, then $\beta'_i \in \mathcal{W}(\Phi, \xi')$ and they are attached to the same axis. Moreover, if β_i, β_j are attached to the same axis and $\beta_i < \beta_j$, then $\beta'_i < \beta'_j$. Note that there are only finitely many classes for the relation of similarity.

Characterization of the classes

We are able now to establish the classes of Φ-topological equivalence in our set of vector fields. Theorem 1 is a consequence of the following result:

Theorem 3. *Two germs of (Φ, H)-vector fields ξ and ξ' are Φ-topologically equivalent if and only if:*

(i) *The points $\mathcal{Q}(\xi, \Phi)$, $\mathcal{Q}(\xi', \Phi)$ coincide (if they exist) and are equally orientated.*
(ii) *The sets $\mathcal{W}(\Phi, \xi)$ and $\mathcal{W}(\Phi, \xi')$ are similar.*
(iii) *The saddle-connections for ξ and ξ' are the same ones.*

Sketch of proof. The direct implication is evident by construction. To show the other one we work by induction on the length of Φ by performing a first blow-up. In order to guarantee the compatibility of the topological equivalences obtained at the induction step, we look for those preserving a *pasting* homeomorphism h between transversal sections to one of the axis. First, we have to give this h in a convenient way: it has to preserve the set of (Φ, ξ)-weights attached to the corresponding axis (see [1–3]). Note also that the conditions on the statement imply that the origin is a singularity of the same type and the same orientation both for ξ and ξ'. If $n = 0$, the difficulty is to find a topological equivalence preserving an h between transversal sections to the one-dimensional invariant variety of a saddle. As explained in [1], it is enough to take this h preserving the respective intrinsic weights α and α' of the origin. This means that if $\pi_\alpha : [0, 1]^2 \to \mathbb{D}_{++}$ is the weighted blow-up of the first quadrant: $\pi_\alpha(r, t) = (r \cos \pi t/2, r^\alpha \sin \pi t/2)$, then h lifts by π_α, $\pi_{\alpha'}$. The induction step works by writing $\Phi = \pi_1 \circ \Phi_1 \circ \Phi_2 \circ \Phi_3$, where π_1 is a blow-up at the origin, and each Φ_i is a sequence of blow-ups with length $n_i < n$ starting at P_i (the origin of the i-chart on M_1). By construction, it holds that $\mathcal{W}(\Phi_i, \xi_1)$ and $\mathcal{W}(\Phi_i, \xi'_1)$ are similar and the saddle-connections for (Φ_i, ξ_1) and (Φ_i, ξ'_1) are the same ones (some weights at $\mathcal{W}(\Phi_i, \xi_1)$ could come from saddle-connections on D lost at the induction step). Also, the positions and orientations of $\mathcal{Q}(\xi, \Phi)$, $\mathcal{Q}(\xi', \Phi)$ (if they exist) do not change. We can place pasting homeomorphisms in suitable axes and, by induction hypothesis, we get Φ_i-topological equivalences H_i, $i = 1, 2, 3$, that can be glued to obtain the desired one. □

Remark. Given a sequence of blow-ups Φ, there are always vector fields ξ, ξ' with non-proportional linear parts that are Φ-topologically equivalent. However, ξ and ξ' are Φ-topologically equivalent for any Φ if and only if their linear parts are proportional.

We think that it is possible to generalize our result to higher dimension by using similar arguments although the combinatorial description is much more complicated.

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