C. R. Acad. Sci. Paris, Ser. I 348 (2010) 553–558

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Harmonic Analysis/Functional Analysis

Functions of perturbed normal operators

Fonctions d'opérateurs perturbés normaux

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ARTICLE INFO

Article history:
Received 23 March 2010
Accepted 6 April 2010
Available online 24 April 2010

Presented by Gilles Pisier

ABSTRACT

In Peller (1985, 1990) [10,11], Aleksandrov and Peller (2009, 2010, 2010) [1–3] sharp estimates for $f(A) - f(B)$ were obtained for self-adjoint operators A and B and for various classes of functions f on the real line $\mathbb{R}$. In this Note we extend those results to the case of functions of normal operators. We show that if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R}^2)$, $0 < \alpha < 1$, of functions of two variables, and N_1 and N_2 are normal operators, then $\|f(N_1) - f(N_2)\| \leqslant \mathrm{const}\,\|f\|_{\Lambda_\alpha}\|N_1 - N_2\|^\alpha$. We obtain a more general result for functions in the space $\Lambda_\omega(\mathbb{R}^2) = \{f\colon |f(\zeta_1) - f(\zeta_2)| \leqslant \mathrm{const}\,\omega(|\zeta_1 - \zeta_2|)\}$ for an arbitrary modulus of continuity ω. We prove that if f belongs to the Besov class $B^1_{\infty 1}(\mathbb{R}^2)$, then it is operator Lipschitz, i.e., $\|f(N_1) - f(N_2)\| \leqslant \mathrm{const}\,\|f\|_{B^1_{\infty 1}}\|N_1 - N_2\|$. We also study properties of $f(N_1) - f(N_2)$ in the case when $f \in \Lambda_\alpha(\mathbb{R}^2)$ and $N_1 - N_2$ belongs to the Schatten–von Neumann class $\boldsymbol{S}_p$.

RÉSUMÉ

On a obtenu dans Peller (1985, 1990) [10,11], Aleksandrov et Peller (2009, 2010, 2010) [1–3] des estimations précises de $f(A) - f(B)$, où A et B sont des opérateurs autoadjoints et f est une fonction sur la droite réelle $\mathbb{R}$. Dans cette note nous obtenons des généralisations de ces résultats pour les opérateurs normaux et pour les fonctions f de deux variables. Nous démontrons que si f appartient à l'espace de Hölder $\Lambda_\alpha(\mathbb{R}^2)$, $0 < \alpha < 1$, alors $\|f(N_1) - f(N_2)\| \leqslant \mathrm{const}\,\|f\|_{\Lambda_\alpha}\|N_1 - N_2\|^\alpha$ pour tous opérateurs normaux N_1 et N_2. Nous obtenons aussi un résultat plus général pour les fonctions de la classe $\Lambda_\omega(\mathbb{R}^2) = \{f\colon |f(\zeta_1) - f(\zeta_2)| \leqslant \mathrm{const}\,\omega(|\zeta_1 - \zeta_2|)\}$. Nous montrons que si f appartient à l'espace de Besov $B^1_{\infty 1}(\mathbb{R}^2)$, alors f est une fonction lipschitzienne opératorielle, c'est-à-dire $\|f(N_1) - f(N_2)\| \leqslant \mathrm{const}\,\|f\|_{B^1_{\infty 1}}\|N_1 - N_2\|$ pour tous opérateurs normaux N_1 et N_2. Nous étudions aussi les propriétés de $f(N_1) - f(N_2)$ quand $f \in \Lambda_\alpha(\mathbb{R}^2)$ et N_1 et N_2 sont des opérateurs normaux tells que $N_1 - N_2$ appartient à l'espace $\boldsymbol{S}_p$ de Schatten–von Neumann.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.04.015

Version française abrégée

Il est bien connu (voir [7]) qu'il y a des fonctions f lipschitziennes sur la droite réelle $\mathbb{R}$ qui ne sont pas *lipschitziennes opératorielles*, c'est-à-dire la condition

$$\left|f(x)-f(y)\right|\leqslant \text{const}\,|x-y|,\quad x,y\in\mathbb{R},$$

n'implique pas que pour tous les opérateurs auto-adjoints A et B l'inégalité

$$\left\|f(A)-f(B)\right\|\leqslant \text{const}\,\|A-B\|$$

soit vraie.

Dans [10] et [11] des conditions nécessaires et des conditions suffisantes sont données pour qu'une fonction f soit lipschitzienne opératorielle. En particulier, il est démontré dans [10] que pour qu'une fonction f soit lipschitzienne opératorielle il est nécessaire que f appartienne localement à l'espace de Besov $B^1_{11}(\mathbb{R})$. Cela implique aussi qu'une fonction lipschitzienne n'est pas nécessairement lipschitzienne opératorielle.

D'autre part il est démontré dans [10] et [11] que si f appartient à l'espace de Besov $B^1_{\infty 1}(\mathbb{R})$, alors f est lipschitzienne opératorielle.

Il se trouve que la situation change dramatiquement si l'on considère les fonctions de la classe $\Lambda_\alpha(\mathbb{R})$ de Hölder d'ordre α, $0<\alpha<1$. Il est démontré dans [1] et [2] que si f appartient à $\Lambda_\alpha(\mathbb{R})$, $0<\alpha<1$ (c'est-à-dire $|f(x)-f(y)|\leqslant \text{const}\,|x-y|^\alpha$), alors f doit être *hölderienne opératorielle d'ordre* α, c'est-à-dire

$$\left\|f(A)-f(B)\right\|\leqslant \text{const}\,\|A-B\|^\alpha.$$

Dans [1] et [2] un problème plus général est considéré pour les fonctions dans l'espace

$$\Lambda_\omega(\mathbb{R})=\left\{f\colon \left|f(\zeta_1)-f(\zeta_2)\right|\leqslant \text{const}\,\omega\big(|\zeta_1-\zeta_2|\big)\right\},$$

où ω est un module de continuité arbitraire.

Finalement il est démontré dans [1] et [3] que si A et B sont des opérateurs autoadjoints tels que $A-B$ appartient à la classe de Schatten–von Neumann $\boldsymbol{S}_p$, $p>1$, et $f\in\Lambda_\alpha(\mathbb{R})$, $0<\alpha<1$, alors $f(A)-f(B)\in\boldsymbol{S}_{p/\alpha}$.

Dans cette Note nous généralisons les résultats ci-dessus aux cas des opérateurs normaux (pas nécessairement bornés).

Nos résultats sont basés sur l'inégalité suivante :

$$\left\|f(N_1)-f(N_2)\right\|\leqslant \text{const}\,\sigma\|f\|_{L^\infty}\|N_1-N_2\|, \tag{1}$$

où f est une fonction bornée sur $\mathbb{R}^2$ dont la transformée de Fourier a un support dans le disque $\{(x,y)\in\mathbb{R}^2\colon x^2+y^2\leqslant\sigma^2\}$ et N_1 et N_2 sont des opérateurs normaux.

Pour établir (1), nous utilisons les intégrales doubles opératorielles et nous démontrons la formule suivante :

$$\begin{aligned}f(N_1)-f(N_2)={}&\iint\limits_{\mathbb{C}^2}\frac{f(x_1,y_1)-f(x_1,y_2)}{y_1-y_2}\,\mathrm{d}E_1(z_1)(B_1-B_2)\,\mathrm{d}E_2(z_2)\\&+\iint\limits_{\mathbb{C}^2}\frac{f(x_1,y_2)-f(x_2,y_2)}{x_1-x_2}\,\mathrm{d}E_1(z_1)(A_1-A_2)\,\mathrm{d}E_2(z_2),\end{aligned}$$

où N_1 et N_2 sont des opérateurs normaux dont la différence est bornée, E_1 et E_2 sont les mesures spéctrales d'N_1 et d'N_2. Ici $x_j=\operatorname{Re}z_j$, $y_j=\operatorname{Im}z_j$, $A_j=\operatorname{Re}N_j$, $B_j=\operatorname{Im}N_j$, $j=1,2$.

En utilisant (1) nous démontrons que si f appartient à la classe de Besov $B^1_{\infty 1}(\mathbb{R}^2)$, alors f est une fonction lipschitzienne opératorielle et

$$\left\|f(N_1)-f(N_2)\right\|\leqslant \text{const}\,\|f\|_{B^1_{\infty 1}(\mathbb{R}^2)}\|N_1-N_2\|$$

pour tous opérateurs normaux N_1 et N_2.

D'autre part, si f appartient à la classe de Hölder $\Lambda_\alpha(\mathbb{R}^2)$, $0<\alpha<1$, alors

$$\left\|f(N_1)-f(N_2)\right\|\leqslant \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\|N_1-N_2\|^\alpha$$

pour tous opérateurs normaux N_1 et N_2.

Supposons maintenant que ω est un module de continuité et

$$\omega_*(x)=x\int_x^\infty\frac{\omega(t)}{t^2}\,\mathrm{d}t,\quad x>0,$$

alors

$$\big\|f(N_1)-f(N_2)\big\| \leqslant \text{const}\,\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega_*\big(\|N_1-N_2\|\big)$$

pour tous opérateurs normaux N_1 et N_2.

Finalement nous démontrons que si $f \in \Lambda_\alpha(\mathbb{R}^2)$, $0<\alpha<1$, et N_1 et N_2 sont des opérateurs normaux dont la différence appartient à l'espace de Schatten–von Neumann $\boldsymbol{S}_p$, $p>1$, alors $f(N_1)-f(N_2)\in\boldsymbol{S}_{p/\alpha}$ et

$$\big\|f(N_1)-f(N_2)\big\|_{\boldsymbol{S}_{p/\alpha}} \leqslant \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\|N_1-N_2\|^\alpha_{\boldsymbol{S}_p}.$$

1. Introduction

In this Note we generalize results of the papers [10,11,1,2], and [3] to the case of normal operators.

A Lipschitz function f on the real line $\mathbb{R}$ (i.e., a function satisfying the inequality $|f(x)-f(y)|\leqslant \text{const}\,|x-y|$, $x,y\in\mathbb{R}$) does not have to be *operator Lipschitz*, i.e.,

$$\big\|f(A)-f(B)\big\| \leqslant \text{const}\,\|A-B\|$$

for arbitrary self-adjoint operators A and B on Hilbert space. The existence of such functions was proved in [7]. Later in [10] and [11] necessary conditions were found for a function f to be operator Lipschitz. In particular, it was shown in [10] that if f is operator Lipschitz, then f belongs locally to the Besov space $B^1_{11}(\mathbb{R})$. This also implies that Lipschitz functions do not have to be operator Lipschitz. Note that in [10] and [11] stronger necessary conditions are also obtained. Note also that the necessary conditions obtained in [10] and [11] are based on the trace class criterion for Hankel operators, see [12, Ch. 6].

On the other hand, it was shown in [10] and [11] that if f belongs to the Besov class $B^1_{\infty 1}(\mathbb{R})$, then f is operator Lipschitz. We refer the reader to [9] for information on Besov spaces.

It was shown in [1] and [2] that the situation dramatically changes if we consider Hölder classes $\Lambda_\alpha(\mathbb{R})$ with $0<\alpha<1$. In this case such functions are necessarily *operator Hölder of order* α, i.e., the condition $|f(x)-f(y)|\leqslant \text{const}\,|x-y|^\alpha$, $x,y\in\mathbb{R}$, implies that for self-adjoint operators A and B on Hilbert space,

$$\big\|f(A)-f(B)\big\| \leqslant \text{const}\,\|A-B\|^\alpha.$$

Note that another proof of this result was found in [8].

This result was generalized in [1] and [2] to the case of functions of class $\Lambda_\omega(\mathbb{R})$ for arbitrary moduli of continuity ω. This class consists of functions f on $\mathbb{R}$, for which $|f(x)-f(y)|\leqslant \text{const}\,\omega(|x-y|)$, $x,y\in\mathbb{R}$.

Finally, we mention here that in [3] properties of operators $f(A)-f(B)$ were studied for functions f in $\Lambda_\alpha(\mathbb{R})$ and self-adjoint operators A and B whose difference $A-B$ belongs to Schatten–von Neumann classes $\boldsymbol{S}_p$.

In this paper we generalize the above results to the case of (not necessarily bounded) normal operators. Throughout the paper we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$.

2. Double operator integrals and the key inequality

Our results are based on the following inequality:

Theorem 2.1. *Let f be a bounded function of class $L^\infty(\mathbb{R}^2)$ whose Fourier transform is supported on the disc $\{\zeta\in\mathbb{C}\colon |\zeta|\leqslant\sigma\}$. Then*

$$\big\|f(N_1)-f(N_2)\big\| \leqslant \text{const}\,\sigma\,\|N_1-N_2\|$$

for arbitrary normal operators N_1 and N_2 with bounded difference.

To prove Theorem 2.1, we obtain a formula for $f(N_1)-f(N_2)$ in terms of double operator integrals. The theory of double operator integrals was developed in [4,5], and [6]. If E_1 and E_2 are spectral measures on $\mathcal{X}_1$ and $\mathcal{X}_2$ and Φ is a bounded measurable function on $\mathcal{X}_1\times\mathcal{X}_2$, then the double operator integral

$$\iint\limits_{\mathcal{X}_1\times\mathcal{X}_2}\Phi(s_1,s_2)\,\mathrm{d}E_1(s_1)T\,\mathrm{d}E_2(s_2)$$

is well defined for all operators T of Hilbert–Schmidt class $\boldsymbol{S}_2$ and determines an operator of class $\boldsymbol{S}_2$. For certain functions Φ the transformer $T\mapsto\iint\Phi\,\mathrm{d}E_1T\,\mathrm{d}E_2$ maps the trace class $\boldsymbol{S}_1$ into itself. For such functions Φ one can define by duality double operator integrals for all bounded operators T. Such functions Φ are called *Schur multipliers* (with respect to the spectral measures E_1 and E_2). We refer the reader to [10] for characterizations of Schur multipliers.

In the following theorem E_1 and E_2 are the spectral measures of normal operators N_1 and N_2. We use the notation $x_j=\operatorname{Re} z_j$, $y_j=\operatorname{Im} z_j$, $A_j=\operatorname{Re} N_j$, $B_j=\operatorname{Im} N_j$, $j=1,2$.

Theorem 2.2. *Let N_1 and N_2 be normal operators such that $N_1 - N_2$ is bounded. Suppose that f is a function in $L^\infty(\mathbb{R}^2)$ such that its Fourier transform $\mathcal{F}f$ has compact support. Then the functions*

$$(z_1, z_2) \mapsto \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2} \quad \text{and} \quad (z_1, z_2) \mapsto \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2}$$

are Schur multipliers with respect to E_1 and E_2 and

$$f(N_1) - f(N_2) = \iint\limits_{\mathbb{C}^2} \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2} \, \mathrm{d}E_1(z_1)(B_1 - B_2) \, \mathrm{d}E_2(z_2) + \iint\limits_{\mathbb{C}^2} \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2} \, \mathrm{d}E_1(z_1)(A_1 - A_2) \, \mathrm{d}E_2(z_2). \tag{2}$$

3. Operator Lipschitz functions of two variables

A continuous function f on $\mathbb{R}^2$ is called *operator Lipschitz* if

$$\|f(N_1) - f(N_2)\| \leqslant \text{const} \, \|N_1 - N_2\|$$

for arbitrary normal operators N_1 and N_2 whose difference is a bounded operator.

Theorem 3.1. *Let f belong to the Besov space $B^1_{\infty 1}(\mathbb{R}^2)$ and let N_1 and N_2 be normal operators whose difference is a bounded operator. Then* (2) *holds and*

$$\|f(N_1) - f(N_2)\| \leqslant \text{const} \, \|f\|_{B^1_{\infty 1}(\mathbb{R}^2)} \|N_1 - N_2\|.$$

In other words, functions in $B^1_{\infty 1}(\mathbb{R}^2)$ must be operator Lipschitz.

As in the case of functions on $\mathbb{R}$, not all Lipschitz functions are operator Lipschitz. In particular, it follows from [10] that if f is an operator Lipschitz function on $\mathbb{R}^2$, then the restriction of f to an arbitrary line belongs locally to the Besov space B^1_{11}.

The next result shows that functions in $B^1_{\infty 1}(\mathbb{R}^2)$ respect trace class perturbations.

Theorem 3.2. *Let f belong to the Besov space $B^1_{\infty 1}(\mathbb{R}^2)$ and let N_1 and N_2 be normal operators such that $N_1 - N_2 \in \boldsymbol{S}_1$. Then $f(N_1) - f(N_2) \in \boldsymbol{S}_1$ and*

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S}_1} \leqslant \text{const} \, \|f\|_{B^1_{\infty 1}(\mathbb{R}^2)} \|N_1 - N_2\|_{\boldsymbol{S}_1}.$$

4. Operator Hölder functions and arbitrary moduli of continuity

For $\alpha \in (0, 1)$, we consider the class $\Lambda_\alpha(\mathbb{R}^2)$ of Hölder functions of order α:

$$\Lambda_\alpha(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{ f\colon \|f\|_{\Lambda_\alpha(\mathbb{R}^2)} = \sup_{z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} < \infty \right\}.$$

The following result shows that in contrast with the class of Lipschitz functions, a Hölder function of order $\alpha \in (0, 1)$ must be *operator Hölder of order α*.

Theorem 4.1. *There exists a positive number c such that for every $\alpha \in (0, 1)$ and every $f \in \Lambda_\alpha(\mathbb{R}^2)$,*

$$\|f(N_1) - f(N_2)\| \leqslant c(1 - \alpha)^{-1} \|f\|_{\Lambda_\alpha(\mathbb{R}^2)} \|N_1 - N_2\|^\alpha$$

for arbitrary normal operators N_1 and N_2.

Consider now more general classes of functions. Let ω be a modulus of continuity. We define the class $\Lambda_\omega(\mathbb{R}^2)$ by

$$\Lambda_\omega(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{ f\colon \|f\|_{\Lambda_\omega(\mathbb{R}^2)} = \sup_{z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} < \infty \right\}.$$

As in the case of functions of one variable (see [1,2]), we define the function ω_* by

$$\omega_*(x) \stackrel{\text{def}}{=} x \int_x^\infty \frac{\omega(t)}{t^2} \, \mathrm{d}t, \quad x > 0.$$

Theorem 4.2. *There exists a positive number c such that for every modulus of continuity ω and every $f \in \Lambda_\omega(\mathbb{R}^2)$,*

$$\big\| f(N_1) - f(N_2) \big\| \leqslant c \| f \|_{\Lambda_\omega(\mathbb{R}^2)} \omega_*\big(\|N_1 - N_2\|\big)$$

for arbitrary normal operators N_1 and N_2.

Corollary 4.3. *Let ω be a modulus of continuity such that*

$$\omega_*(x) \leqslant \operatorname{const} \omega(x), \quad x > 0,$$

and let $f \in \Lambda_\omega(\mathbb{R}^2)$. Then

$$\big\| f(N_1) - f(N_2) \big\| \leqslant \operatorname{const} \| f \|_{\Lambda_\omega(\mathbb{R}^2)} \omega\big(\|N_1 - N_2\|\big)$$

for arbitrary normal operators N_1 and N_2.

5. Perturbations of class $\boldsymbol{S}_p$

In this section we study properties of $f(N_1) - f(N_2)$ in the case when $f \in \Lambda_\alpha(\mathbb{R}^2)$, $0 < \alpha < 1$, and N_1 and N_2 are normal operators such that $N_1 - N_2$ belongs to the Schatten–von Neumann class $\boldsymbol{S}_p$. The following theorem generalizes Theorem 5.8 of [3] to the case of normal operators.

Theorem 5.1. *Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$ and for arbitrary normal operators N_1 and N_2 with $N_1 - N_2 \in \boldsymbol{S}_p$, the operator $f(N_1) - f(N_2)$ belongs to $\boldsymbol{S}_{p/\alpha}$ and the following inequality holds:*

$$\big\| f(N_1) - f(N_2) \big\|_{\boldsymbol{S}_{p/\alpha}} \leqslant c \| f \|_{\Lambda_\alpha(\mathbb{R}^2)} \|N_1 - N_2\|_{\boldsymbol{S}_p}^\alpha.$$

For $p = 1$ this is not true even for self-adjoint operators, see [3]. Note that the construction of the counterexample in [3] involves Hankel operators and is based on the criterion of membership of $\boldsymbol{S}_p$ for Hankel operators, see [12, Ch. 6].

The following weak version of Theorem 5.1 holds:

Theorem 5.2. *Let $0 < \alpha < 1$ and let $f \in \Lambda_\alpha(\mathbb{R}^2)$. Suppose that N_1 and N_2 are normal operators such that $N_1 - N_2 \in \boldsymbol{S}_1$. Then $f(N_1) - f(N_2) \in \boldsymbol{S}_{\frac{1}{\alpha},\infty}$, i.e.,*

$$s_j\big(f(N_1) - f(N_2)\big) \leqslant \operatorname{const} \| f \|_{\Lambda_\alpha(\mathbb{R}^2)} (1 + j)^{-\alpha}, \quad j \geqslant 0.$$

Here $s_j(T)$ is the jth singular value of a bounded operator T.

On the other hand, the conclusion of Theorem 5.1 remains valid even for $p = 1$ if we impose a slightly stronger assumption on f.

Theorem 5.3. *Let $0 < \alpha < 1$ and let f belong to the Besov space $B^\alpha_{\infty 1}(\mathbb{R}^2)$. Suppose that N_1 and N_2 are normal operators such that $N_1 - N_2 \in \boldsymbol{S}_1$. Then $f(N_1) - f(N_2) \in \boldsymbol{S}_{1/\alpha}$ and*

$$\big\| f(N_1) - f(N_2) \big\|_{\boldsymbol{S}_{1/\alpha}} \leqslant \operatorname{const} \| f \|_{B^\alpha_{\infty 1}(\mathbb{R}^2)} \|N_1 - N_2\|_{\boldsymbol{S}_1}^\alpha.$$

We conclude this section with the following improvement of Theorem 5.1.

Theorem 5.4. *Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$, every $l \in \mathbb{Z}_+$, and arbitrary normal operators N_1 and N_2 with bounded $N_1 - N_2$, the following inequality holds:*

$$\sum_{j=0}^{l} \big(s_j\big(\big|f(N_1) - f(N_2)\big|^{1/\alpha}\big)\big)^p \leqslant c \| f \|_{\Lambda_\alpha(\mathbb{R}^2)}^{p/\alpha} \sum_{j=0}^{l} \big(s_j(N_1 - N_2)\big)^p.$$

6. Commutators and quasicommutators

We can generalize all the results listed above to the case of quasicommutators $f(N_1)Q - Qf(N_2)$ and obtain estimates for such quasicommutators in terms of $N_1 Q - Q N_2$ and $N_1^* Q - Q N_2^*$. Here Q is a bounded operator. In particular, if Q is the identity operator, we arrive at the problem of estimating operator differences $f(N_1) - f(N_2)$ that has been considered above. On the other hand, if $N_1 = N_2 = N$, we arrive at the problem of estimating commutators $f(N)Q - Qf(N)$ in terms of $NQ - QN$ and $N^*Q - QN^*$.

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