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Harmonic Analysis/Analytic Geometry

Buffon needle lands in ϵ-neighborhood of a 1-dimensional Sierpinski Gasket with probability at most $|\log \epsilon|^{-c}$

Une estimation de la probabilité pour l'aiguille de Buffon de se situer dans un ϵ-voisinage de l'ensemble de Sierpinski

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ABSTRACT

In recent years, relatively sharp quantitative results in the spirit of the Besicovitch projection theorem have been obtained for self-similar sets by studying the L^p norms of the "projection multiplicity" functions, f_θ, where $f_\theta(x)$ is the number of connected components of the partial fractal set that orthogonally project in the θ direction to cover x. In Nazarov et al. (2008) [4], it was shown that n-th partial 4-corner Cantor set with self-similar scaling factor 1/4 decays in Favard length at least as fast as $\frac{C}{n^p}$, for $p < 1/6$. In Bond and Volberg (2009) [1], this same estimate was proved for the 1-dimensional Sierpinski gasket for some $p > 0$. A few observations were needed to adapt the approach of Nazarov et al. (2008) [4] to the gasket: we sketch them here. We also formulate a result about all self-similar sets of dimension 1.

RÉSUMÉ

On donne une estimation de la probabilité pour que l'aiguille de Buffon soit ϵ-proche d'un ensemble de Cantor–Sierpinski. On trouve une majoration de cette probabilité en $|\log \epsilon|^{-c}$, où c est une constante strictement positive, cette constante n'est pas connue de mannière précise, mais l'estimation est optimale.

1. Definitions and result

Let $E \subset \mathbb{C}$, and let proj_θ denote orthogonal projection onto the line having angle θ with the real axis. The *average projected length* or *Favard length* of E, $\text{Fav}(E)$, is given by,

$$\text{Fav}(E) = \frac{1}{\pi} \int_0^\pi \left|\text{proj}_\theta(E)\right| \, d\theta.$$

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For bounded sets, Favard length is also called *Buffon needle probability*, since up to a normalization constant, it is the likelihood that a long needle dropped with independent, uniformly distributed orientation and distance from the origin will intersect the set somewhere.

Set $B(z_0, r) := \{z \in \mathbb{C}\colon\ |z - z_0| < r\}$. For $\alpha \in \{-1, 0, 1\}^n$, let

$$z_\alpha := \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k e^{i\pi[\frac{1}{2}+\frac{2}{3}\alpha_k]}, \qquad \mathcal{G}_n := \bigcup_{\alpha\in\{-1,0,1\}^n} B\big(z_\alpha, 3^{-n}\big).$$

This set is our approximation of a partial Sierpinski gasket; it is strictly larger. We may still speak of the approximating discs as "Sierpinski triangles."

The main result is:

Theorem 1.1. $\mathrm{Fav}(\mathcal{G}_n) \leqslant \frac{C}{n^{1/14}}$.

The set $\mathcal{G}_n$ is a 3^{-n} approximation to the Besicovitch irregular set (see [2] for definition) called Sierpinski gasket. Recently one detects a considerable interest in estimating the Favard length of such ϵ-neighborhoods of Besicovitch irregular sets, see [5,6,4,3]. In [5] a random model of such Cantor set is considered and the estimate $\lesssim \frac{1}{n}$ infinitely often, almost surely is proved. But for non-random self-similar sets the estimates of [5] are more in terms of $\frac{1}{\log\ldots\log n}$ (number of logarithms depending on n) and more suitable for general class of "quantitatively Besicovitch irregular sets" treated in [6].

Let $f_{n,\theta} := \frac{1}{2}\nu_n * 3^n\chi_{[-3^{-n},3^{-n}]}$, where

$$\nu_n := *_{k=1}^{n}\widetilde{\nu}_k \quad \text{and} \quad \widetilde{\nu}_k := \frac{1}{3}[\delta_{3^{-k}\cos(\pi/2-\theta)} + \delta_{3^{-k}\cos(-\pi/6-\theta)} + \delta_{3^{-k}\cos(7\pi/6-\theta)}].$$

For $K > 0$, let $A_K := A_{K,n,\theta} := \{x\colon\ f_{n,\theta} \geqslant K\}$. Let $\mathcal{L}_{\theta,n} := \mathrm{proj}_\theta(\mathcal{G}_n)$. Notice that $\mathcal{L}_{\theta,n} = A_{1,n,\theta}$. For our result, some maximal versions of these are needed:

$$f^*_{N,\theta} := \max_{n\leqslant N} f_{n,\theta}, \qquad A^*_K := A^*_{K,n,\theta} := \big\{x\colon\ f^*_{n,\theta} \geqslant K\big\}.$$

Also, let $E := E_N := \{\theta\colon\ |A^*_K| \leqslant K^{-3}\}$ for $K = N^{\epsilon_0}$, ϵ_0.

Later, we will jump to the Fourier side, where the function

$$\varphi_\theta(x) := \frac{1}{3}\big[e^{-i\cos(\pi/2-\theta)x} + e^{-i\cos(-\pi/6-\theta)x} + e^{-i\cos(7\pi/6-\theta)x}\big]$$

plays the central role: set $\widehat{\nu_n}(x) = \prod_{k=1}^{n}\varphi_\theta(3^{-k}x)$.

2. General philosophy

Fix θ. If the mass of $f_{n,\theta}$ is concentrated on a small set, then $\|f_{n,\theta}\|_p$ should be large for $p > 1$ – and vice versa. $1 = \int f \leqslant \|f_{n,\theta}\|_p\|\chi_{\mathcal{L}_{\theta,n}}\|_q$, so $m(\mathcal{L}_{\theta,n}) \geqslant \|f\|_p^{-q}$, a decent estimate. The other basic estimate is not so sharp:

$$m(\mathcal{L}_{\theta,N}) \leqslant 1 - (K-1)m(A_{K,N,\theta}). \tag{1}$$

However, a combinatorial self-similarity argument of [4] and revisited in [1] shows that for the Favard length problem, it bootstraps well under further iterations of the similarity maps:

Theorem 2.1. *If $\theta \notin E_N$, then $|\mathcal{L}_{\theta,NK^3}| \leqslant \frac{C}{K}$.*

Note that the maximal version f^*_N is used here. A stack of K triangles at stage n generally accounts for more stacking per step the smaller n is. For fixed $x \in A^*_{K,N,\theta}$, the above theorem considers the smallest n such that $x \in A_{K,n,\theta}$, and uses self-similarity and the Hardy–Littlewood theorem to prove its claim by successively refining an estimate in the spirit of (1). Of course, now Theorem 1.1 follows from the following:

Theorem 2.2. *Let $\epsilon_0 < 1/11$. Then for $N \gg 1$, $|E_N| < N^{-\epsilon_0}$.*

It turns out that L^2 theory on the Fourier side is of great use here. It is proved in [4,1]:

Theorem 2.3. *For all $\theta \in E_N$ and for all $n \leqslant N$, $\|f_{n,\theta}\|^2_{L^2} \leqslant CK$.*

One can then take small sample integrals on the Fourier side and look for lower bounds as well. Let $K = N^{\epsilon_0}$, and let $m = 2\epsilon_0 \log_3 N$. Theorem 2.3 easily implies the existence of $\tilde{E} \subset E$ such that $|\tilde{E}| > |E/2|$ and number n, $N/4 < n < N/2$, such that for all $\theta \in \tilde{E}$,

$$\int_{3^{n-m}}^{3^n} \prod_{k=0}^{n} \left|\varphi_\theta\left(3^{-k}x\right)\right|^2 \mathrm{d}x \leqslant \frac{2CKm}{N} \leqslant 2\epsilon_0 N^{\epsilon_0 - 1} \log N.$$

Number n does not depend on θ; n can be chosen to satisfy the estimate in the average over $\theta \in E$, and then one chooses $\tilde{E}$. Let $I := [3^{n-m}, 3^n]$.

Now the main result amounts to this (with absolute constant A large enough):

Theorem 2.4.

$$\theta \in \tilde{E}: \quad \int_I \prod_{k=0}^{n} \left|\varphi_\theta\left(3^{-k}x\right)\right|^2 \mathrm{d}x \geqslant c3^{m-2\cdot Am} = cN^{-2\epsilon_0(2A-1)}.$$

The result: $2\epsilon_0 \log N \geqslant N^{1-\epsilon_0(4A-1)}$, i.e., $N \leqslant N^*$. Now we sketch the proof of Theorem 2.4. We split up the product into two parts: high and low-frequency: $P_{1,\theta}(z) = \prod_{k=0}^{n-m-1} \varphi_\theta(3^{-k}z)$, $P_{2,\theta}(z) = \prod_{k=n-m}^{n} \varphi_\theta(3^{-k}z)$.

Theorem 2.5. *For all* $\theta \in E$, $\int_I |P_{1,\theta}|^2 \,\mathrm{d}x \geqslant C3^m$.

Low frequency terms do not have as much regularity, so we must control the damage caused by the *set of small values*, $SSV(\theta) := \{x \in I\colon\ |P_2(x)| \leqslant 3^{-\ell}\}$, $\ell = \alpha m$ with sufficiently large constant α. In the next result we claim the existence of $\mathcal{E} \subset \tilde{E}$, $|\mathcal{E}| > |\tilde{E}/2|$ with the following property:

Theorem 2.6.

$$\int_{\tilde{E}} \int_{SSV(\theta)} \left|P_{1,\theta}(x)\right|^2 \mathrm{d}x\,\mathrm{d}\theta \leqslant 3^{2m-\ell/2} \quad \Rightarrow \quad \forall \theta \in \mathcal{E}, \quad \int_{SSV(\theta)} \left|P_{1,\theta}(x)\right|^2 \mathrm{d}x \leqslant cK3^{2m-\ell/2}.$$

Then Theorems 2.5 and 2.6 give Theorem 2.4.

3. Locating the zeros of P_2

We can consider $\Phi(x, y) = 1 + e^{ix} + e^{iy}$. The key observations are

$$\left|\Phi(x, y)\right|^2 \geqslant a\left(\left|4\cos^2 x - 1\right|^2 + \left|4\cos^2 y - 1\right|^2\right), \qquad \frac{\sin 3x}{\sin x} = 4\cos^2 x - 1.$$

Changing variable we can replace $3\varphi_\theta(x)$ by $\phi_t(x) = \Phi(x, tx)$. Consider

$$P_{2,t}(x) := \prod_{k=n-m}^{n} \frac{1}{3}\phi_t\left(3^{-k}x\right), \qquad P_{1,t}(x) := \prod_{k=0}^{n-m} \frac{1}{3}\phi_t\left(3^{-k}x\right).$$

We need

$$SSV(t) := \left\{x \in I\colon\ \left|P_{2,t}(x)\right| \leqslant 3^{-\ell}\right\}.$$

One can easily imagine it if one considers

$$\Omega := \left\{(x, y) \in [0, 2\pi]^2\colon\ \left|\mathcal{P}(x, y)\right| := \left|\prod_{k=0}^{m} \Phi\left(3^k x, 3^k y\right)\right| \leqslant 3^{m-\ell}\right\}.$$

Moreover (using that if $x \in SSV(t)$ then $3^{-n}x \geqslant 3^{-m}$, and using $x\,\mathrm{d}x\,\mathrm{d}t = \mathrm{d}x\,\mathrm{d}y$), we change variable in the next integral:

$$\int_{\tilde{E}} \int_{SSV(t)} \left|P_{1,t}(x)\right|^2 \mathrm{d}x\,\mathrm{d}t = 3^{-2n+2m} \cdot 3^n \int_{\tilde{E}} \int_{3^{-n}SSV(t)} \left|\prod_{k=m}^{n} \Phi\left(3^k x, 3^k tx\right)\right|^2 \mathrm{d}x\,\mathrm{d}t$$

$$\leqslant 3^{-n+3m} \int_{\Omega} \left|\prod_{k=m}^{n} \Phi\left(3^k x, 3^k y\right)\right|^2 \mathrm{d}x\,\mathrm{d}y.$$

Now notice that by our key observations

$$\Omega \subset \{(x, y) \in [0, 2\pi]^2 \colon |\sin 3^{m+1} x|^2 + |\sin 3^{m+1} y|^2 \leqslant a^{-m} 3^{2m-2\ell} \leqslant 3^{-\ell}\}.$$

The latter set $\mathcal{Q}$ is the union of $4 \cdot 3^{2m+2}$ squares Q of size $3^{-m-\ell/2} \times 3^{-m-\ell/2}$. Fix such a Q and estimate,

$$\begin{aligned}\int_Q \left|\prod_{k=m}^{n} \Phi(3^k x, 3^k y)\right|^2 \mathrm{d}x\,\mathrm{d}y &\leqslant 3^\ell \int_Q \left|\prod_{k=m+\ell/2}^{n} \Phi(3^k x, 3^k y)\right|^2 \mathrm{d}x\,\mathrm{d}y \\ &\leqslant 3^\ell \cdot (3^{-m-\ell/2})^2 \int_{[0,2\pi]^2} \left|\prod_{k=0}^{n-m-\ell/2} \Phi(3^k x, 3^k y)\right|^2 \mathrm{d}x\,\mathrm{d}y \\ &\leqslant 3^\ell \cdot (3^{-m-\ell/2})^2 \cdot 3^{n-m-\ell/2} \\ &= 3^{-2m} \cdot 3^{n-m-\ell/2}.\end{aligned}$$

Therefore, taking into account the number of squares Q in $\mathcal{Q}$ and the previous estimates we get

$$\int_E \int_{SSV(t)} |P_{1,t}(x)|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant 3^{2m-\ell/2}.$$

Theorem 2.6 is proved.

To prove Theorem 2.5 we need the following simple lemma.

Lemma 3.1. *Let C be large enough. Let $j = 1, 2, \ldots, k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{\alpha_j\}_{j=1}^k$. Suppose*

$$\int_{\mathbb{R}} \left(\sum_{\alpha \in A} \chi_{[\alpha-1,\alpha+1]}(x)\right)^2 \mathrm{d}x \leqslant S. \quad \textit{Then} \int_0^1 \left|\sum_{\alpha\in A} c_\alpha e^{i\alpha y}\right|^2 \mathrm{d}y \leqslant CS.$$

Some key facts useful for its proof:

$$\int_0^1 \left|\sum_{\alpha\in A} c_\alpha e^{i\alpha y}\,\mathrm{d}y\right|^2 \leqslant e \int_0^\infty \left|\sum_{\alpha\in A} c_\alpha e^{i(\alpha+i)y}\,\mathrm{d}y\right|^2 = e \int_{\mathbb{R}} \left|\sum_{\alpha\in A} \frac{c_\alpha}{\alpha + i - x}\right|^2 \mathrm{d}x,$$

and the fact that $H^2(\mathbb{C}_+)$ is orthogonal to $\overline{H^2(\mathbb{C}_+)}$, so one can pass to the Poisson kernel.

4. The general case

Let us have k closed disjoint discs of radii $1/k$ located in the unit disc. We build k^n small discs of radii k^{-n} by iterating k linear maps from small discs onto the unit disc. Call the resulting union $S_k(n)$. We would like to show that exactly as in the case of $k = 3$ considered above and in a very special case of $k = 4$ considered in [4] $\mathrm{Fav}(S_k(n)) \leqslant Cn^{-c}$, $c > 0$. However, presently we can prove only a weaker result.

Theorem 4.1.

$$\mathrm{Fav}(S_k(n)) \leqslant Ce^{-c(\log n)^{1/2}}, \quad c > 0.$$

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