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Topology of character varieties and representations of quivers

Topologie des variétés de caractères et représentations de carquois

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A R T I C L E   I N F O

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A B S T R A C T

In Hausel et al. (2008) [10] we presented a conjecture generalizing the Cauchy formula for Macdonald polynomial. This conjecture encodes the mixed Hodge polynomials of the character varieties of representations of the fundamental group of a punctured Riemann surface of genus g. We proved several results which support this conjecture. Here we announce new results which are consequences of those in Hausel et al. (2008) [10].

R É S U M É

Dans Hausel et al. (2008) [10] nous avons énoncé une conjecture qui généralise la formule de Cauchy pour les polynômes de Macdonald. Cette formule contient l'information sur les polynômes de Hodge mixtes des variétés de représentations du groupe fondamental d'une surface de Riemann épointée de genre g. Nous avons montré plusieurs résultats qui appuient notre conjecture. Ici nous présentons de nouveaux résultats qui sont des conséquences de ceux de Hausel et al. (2008) [10].

Version française abrégée

Soient $g \geqslant 0$ et $k > 0$ deux entiers et $\mathbf{x}_1 = \{x_{1,1}, x_{1,2}, \ldots\}, \ldots, \mathbf{x}_k = \{x_{k,1}, x_{k,2}, \ldots\}$ des ensembles infinis disjoints de variables. On définit [10]

$$\Omega(z,w) = \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda\big(\mathbf{x}_i; z^2, w^2\big).$$

où $\mathcal{P}$ est l'ensemble de toutes les partitions,

$$\mathcal{H}_\lambda(z,w) := \prod \frac{(z^{2a+1} - w^{2l+1})^{2g}}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})}$$

est une (z,w)-déformation de la $(2g-2)$-ième puissance du polynôme de crochet et où $H_\lambda(\mathbf{x}_i; z, w)$ est la fonction symétrique de Macdonald définie dans [6, I.11]. Pour $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$, on pose

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$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) := (z^2-1)(1-w^2)\langle \mathrm{Log}(\Omega(z,w)), h_{\boldsymbol{\mu}}\rangle$$

où $h_\mu := h_{\mu^1}(\mathbf{x}_1)\cdots h_{\mu^k}(\mathbf{x}_k)$ est le produit des fonctions symétriques complètes et $\langle\,,\,\rangle$ est le produit de Hall. Soit $(C_1,\dots,C_k)$ un k-uplet *générique* de classes de conjugaison semisimples de $\mathrm{GL}_n(\mathbb{C})$ de type $\boldsymbol{\mu}$, i.e., les coordonnées μ^i de $\boldsymbol{\mu}$ donnent les multiplicités des valeurs propres de C_i. On définit alors $\mathcal{M}_{\boldsymbol{\mu}}$ comme le quotient affine (GIT)

$$\mathcal{M}_{\boldsymbol{\mu}} := \{A_1,B_1,\dots,A_g,B_g \in \mathrm{GL}_n(\mathbb{C}), X_1\in \mathcal{C}_1,\dots,X_k\in\mathcal{C}_k \mid (A_1,B_1)\cdots(A_g,B_g)X_1\cdots X_k = I_n\}//\mathrm{GL}_n(\mathbb{C}),$$

où (A,B) désigne le commutateur $ABA^{-1}B^{-1}$. Dans [10] on conjecture que $\mathbf{H}_c(\mathcal{M}_{\boldsymbol{\mu}};q,t) := (t\sqrt{q})^{d_{\boldsymbol{\mu}}}\mathbb{H}_{\boldsymbol{\mu}}(-1/\sqrt{q}, t\sqrt{q})$, où $d_{\boldsymbol{\mu}}$ est la dimension de $\mathcal{M}_{\boldsymbol{\mu}}$, coincide avec le polynôme de Hodge mixte $H_c(\mathcal{M}_{\boldsymbol{\mu}};q,t) = \sum_{i,k} h_c^{i,i;k}(\mathcal{M}_{\boldsymbol{\mu}})q^i t^k$ (les $h_c^{i,j;k}$ étant les nombres de Hodge mixtes à support compact). On démontre [loc. cit.] que l'identité est vraie si on spécialise par $(q,t)\mapsto(q,-1)$ auquel cas le polynôme de Hodge devient ce qu'on appelle le *E-polynôme* $E(\mathcal{M}_{\boldsymbol{\mu}};q)$. On s'intéresse aussi à la spécialisation qui revient à prendre la partie pure $\sum_i h_c^{i,i;2i}(\mathcal{M}_{\boldsymbol{\mu}})T^i$ du polynôme de Hodge. On montre dans [loc. cit.] que la partie pure de $\mathbf{H}_c(\mathcal{M}_{\boldsymbol{\mu}};q,t)$ coincide avec la multiplicité du caractère trivial de $\mathrm{GL}_n(\mathbb{F}_q)$ dans un produit tensoriel $\Lambda\otimes R_1\otimes\cdots\otimes R_k$ où $(R_1,\dots,R_k)$ est un k-uplet *générique* de caractères complexes irréductibles *semisimples* de $G_n(\mathbb{F}_q)$ et Λ est un caractère associé au genre g. Le but de ces notes est d'annoncer trois nouveaux résultats qui seront démontrés dans [11].

A partir de la donnée $(\boldsymbol{\mu},g)$ on peut définir un carquois étoilé Γ avec k branches et g lacets sur le sommet central (voir [10]) ainsi qu'un vecteur dimension. On note alors $A_{\boldsymbol{\mu}}(q)$ le nombre de représentations absolument indécomposables de Γ sur un corps fini $\mathbb{F}_q$. On sait depuis [12] que $A_{\boldsymbol{\mu}}(q)$ est un polynôme en q à coefficients entiers. Dans ce même article de Kac il est aussi conjecturé que $A_{\boldsymbol{\mu}}(q)\in\mathbb{Z}_{\geqslant 0}[q]$. Notre premier résultat dit que $A_{\boldsymbol{\mu}}(q)$ coincide avec la partie pure de $\mathbf{H}_c(\mathcal{M}_{\boldsymbol{\mu}};q,t)$ et donc que $A_{\boldsymbol{\mu}}(q) = \langle \Lambda\otimes\bigotimes_{i=1}^k R_i, 1\rangle$. Ainsi notre conjecture $\mathbf{H}_c(\mathcal{M}_{\boldsymbol{\mu}};q,t) = H_c(\mathcal{M}_{\boldsymbol{\mu}};q,t)$ donne une interpretation des coefficients de $A_{\boldsymbol{\mu}}(q)$ en fonction des nombres de Betti et implique donc que la conjecture de Kac est vraie pour les carquois de la forme Γ. Le deuxième résultat dit que $\mathcal{M}_{\boldsymbol{\mu}}$ est connexe. La preuve utilise l'identité $\mathbf{H}_c(\mathcal{M}_{\boldsymbol{\mu}};q,-1) = E(\mathcal{M}_{\boldsymbol{\mu}};q)$. Notre troisième résultat concerne la conjecture $\mathbf{H}_c(\mathcal{M}_{\boldsymbol{\mu}};q,t) = H_c(\mathcal{M}_{\boldsymbol{\mu}};q,t)$ lorsque $g=1$, $k=1$ et $\boldsymbol{\mu}$ est la partition $(n-1,1)$. On montre alors que la conjecture est vraie pour un certain nombre d'entiers n. La preuve utilise les résultats de [8] sur le polynôme de Hodge du schéma de Hilbert $(\mathbb{C}^*\times\mathbb{C}^*)^{[n]}$.

1. Review of the results of [10]

1.1. Cauchy function

Fix integers $g\geqslant 0$ and $k>0$. Let $\mathbf{x}_1 = \{x_{1,1}, x_{1,2},\dots\},\dots,\mathbf{x}_k = \{x_{k,1},x_{k,2},\dots\}$ be k sets of infinitely many independent variables and let Λ be the ring of functions separately symmetric in each set of variables. Let $\mathcal{P}$ be the set of partitions. For $\lambda\in\mathcal{P}$, let $\tilde{H}_\lambda(\mathbf{x}_i;q,t)\in\Lambda\otimes\mathbb{Q}(q,t)$ be the *Macdonald symmetric function* defined in [6, I.11].

Define the *k-point genus g Cauchy function*

$$\Omega(z,w) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(z,w)\prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2),$$

where

$$\mathcal{H}_\lambda(z,w) := \prod \frac{(z^{2a+1}-w^{2l+1})^{2g}}{(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})}$$

is a (z,w)-deformation of the $(2g-2)$-th power of the standard hook polynomial. Let Exp be the plethystic exponential and let Log be its inverse [10, 2.3]. For $\boldsymbol{\mu} = (\mu^1,\dots,\mu^k)\in\mathcal{P}^k$, let

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) := (z^2-1)(1-w^2)\langle \mathrm{Log}(\Omega(z,w)), h_{\boldsymbol{\mu}}\rangle,$$

where $h_\mu := h_{\mu^1}(\mathbf{x}_1)\cdots h_{\mu^k}(\mathbf{x}_k)\in\Lambda$ is the product of the complete symmetric functions and $\langle,\rangle$ is the extended Hall pairing.

1.2. Character and quiver varieties

We let $(\mathcal{C}_1,\dots,\mathcal{C}_k)$ be a *generic* k-tuple of semisimple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ of type $\boldsymbol{\mu}$, i.e., μ^i gives the multiplicities of the eigenvalues of C_i. We consider the GIT quotient

$$\mathcal{M}_{\boldsymbol{\mu}} := \{A_1,B_1,\dots,A_g,B_g \in \mathrm{GL}_n(\mathbb{C}), X_1\in \mathcal{C}_1,\dots,X_k\in\mathcal{C}_k \mid (A_1,B_1)\cdots(A_g,B_g)X_1\cdots X_k = I_n\}//\mathrm{GL}_n(\mathbb{C}),$$

where $(A,B) := ABA^{-1}B^{-1}$. We call $\mathcal{M}_{\boldsymbol{\mu}}$ a *generic* character variety. Let $(\mathcal{O}_1,\dots,\mathcal{O}_k)$ be a generic k-tuple of semisimple adjoint orbits of $\mathfrak{gl}_n(\mathbb{C})$ of type $\boldsymbol{\mu}$. We put

$$\mathcal{Q}_{\boldsymbol{\mu}} := \big\{A_1, B_1, \dots, A_g, B_g \in \mathfrak{gl}_n(\mathbb{C}), C_1 \in \mathcal{O}_1, \dots, C_k \in \mathcal{O}_k \mid [A_1, B_1] + \cdots + [A_g, B_g] + C_1 + \cdots + C_k = O\big\} /\!/ \mathrm{GL}_n(\mathbb{C}).$$

Then $\mathcal{Q}_{\boldsymbol{\mu}}$ is a quiver variety associated to the comet-shaped quiver in Section 2 (see for instance [3] for more details).

In [10], we proved that, if non-empty, $\mathcal{M}_{\boldsymbol{\mu}}$ and $\mathcal{Q}_{\boldsymbol{\mu}}$ are non-singular algebraic varieties of pure dimension

$$d_{\boldsymbol{\mu}} = n^2(2g - 2 + k) - \sum_{i,j} \big(\mu^i_j\big)^2 + 2.$$

Let $H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t) := \sum_{i,j,k} h_c^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}}) x^i y^j t^k$ be the compactly supported mixed Hodge polynomial. It is a common deformation of the compactly supported Poincaré polynomial $P_c(\mathcal{M}_{\boldsymbol{\mu}}; t) = H_c(\mathcal{M}_{\boldsymbol{\mu}}; 1, 1, t)$ and the so-called E-polynomial $E(\mathcal{M}_{\boldsymbol{\mu}}; x, y) = H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, -1)$. We have the following conjecture [10, Conjecture 1.1.1]:

Conjecture 1.1. *The polynomial $H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t)$ depends only on xy and t. If we let $H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t) = H_c(\mathcal{M}_{\boldsymbol{\mu}}; \sqrt{q}, \sqrt{q}, t)$ then*

$$H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t) = (t\sqrt{q})^{d_{\boldsymbol{\mu}}} \mathbb{H}_{\boldsymbol{\mu}}\left(-\frac{1}{\sqrt{q}}, t\sqrt{q}\right). \tag{1}$$

This conjecture implies the following one:

Conjecture 1.2 *(Curious Poincaré duality).*

$$H_c\left(\mathcal{M}_{\boldsymbol{\mu}}; \frac{1}{qt^2}, t\right) = (qt)^{-d_{\boldsymbol{\mu}}} H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t).$$

The two following theorems are proved in [10]:

Theorem 1.3. *The E-polynomial $E(\mathcal{M}_{\boldsymbol{\mu}}; x, y)$ depends only on xy and if we let $E(\mathcal{M}_{\boldsymbol{\mu}}; q) = E(\mathcal{M}_{\boldsymbol{\mu}}; \sqrt{q}, \sqrt{q})$, we have*

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{\frac{1}{2}d_{\boldsymbol{\mu}}} \mathbb{H}_{\boldsymbol{\mu}}\left(\frac{1}{\sqrt{q}}, \sqrt{q}\right). \tag{2}$$

As a corollary we have:

Corollary 1.4. *The E-polynomial is* palindromic, *that is*

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{d_{\boldsymbol{\mu}}} E\big(\mathcal{M}_{\boldsymbol{\mu}}; q^{-1}\big) = \sum_i \left(\sum_k (-1)^k h^{i,i;k}(\mathcal{M}_{\boldsymbol{\mu}})\right) q^i.$$

We say that $\boldsymbol{\mu}$ is *indivisible* if the gcd of all the parts of the partitions $\mu^1, \dots, \mu^k$ is equal to 1. It is possible to choose k generic semisimple adjoint orbits of type $\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is indivisible [10, Lemma 2.2.2].

Theorem 1.5. *For $\boldsymbol{\mu}$ indivisible, the mixed Hodge structure on $H_c^*(\mathcal{Q}_{\boldsymbol{\mu}}, \mathbb{C})$ is pure. If we let $E(\mathcal{Q}_{\boldsymbol{\mu}}; q) = E(\mathcal{Q}_{\boldsymbol{\mu}}; \sqrt{q}, \sqrt{q})$, then*

$$P_c(\mathcal{Q}_{\boldsymbol{\mu}}; \sqrt{q}) = E(\mathcal{Q}_{\boldsymbol{\mu}}; q) = q^{\frac{1}{2}d_{\boldsymbol{\mu}}} \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}). \tag{3}$$

Note that formula (2) is the specialization $t \mapsto -1$ of formula (1). Assuming Conjecture 1.1, formula (3) implies that the $2i$-th Betti number of $\mathcal{Q}_{\boldsymbol{\mu}}$ equals the dimension of the i-th piece of the pure part of the cohomology of $\mathcal{M}_{\boldsymbol{\mu}}$, namely, $h_c^{i,i;2i}(\mathcal{M}_{\boldsymbol{\mu}})$. Furthermore, when $g = 0$, the first author conjectures [9] that there is an isomorphism between the pure part of $H_c^i(\mathcal{M}_{\boldsymbol{\mu}}, \mathbb{C})$ and $H_c^i(\mathcal{Q}_{\boldsymbol{\mu}}, \mathbb{C})$ induced by the Riemann–Hilbert monodromy map $\mathcal{Q}_{\boldsymbol{\mu}} \to \mathcal{M}_{\boldsymbol{\mu}}$. This would give a geometric interpretation of Theorem 1.5 in this case.

1.3. Multiplicities in tensor products

Given $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k) \in \mathcal{P}^k$, we can choose a *generic* k-tuple $(R_1, \dots, R_k)$ of semisimple irreducible complex characters of $\mathrm{GL}_n(\mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite field with q elements [10]. We also denote by $\Lambda : \mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{C}$ the character $h \mapsto q^{g \dim Z(h)}$ where $Z(h)$ is the centralizer of h in $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$. Then we have [10, 6.1.1]:

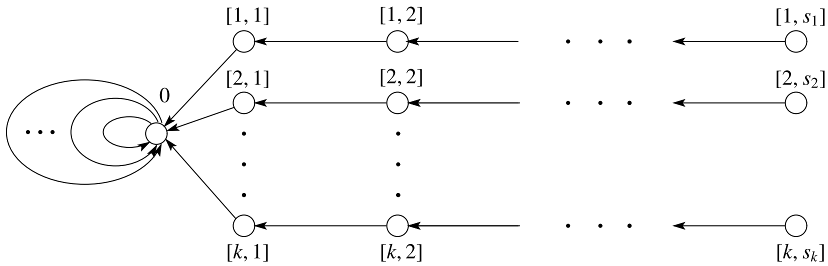
Theorem 1.6.

$$\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}),$$

where $R_{\boldsymbol{\mu}} = \bigotimes_{i=1}^k R_i$.

2. Absolutely indecomposable representations

Let $\mathbf{s} = (s_1, \ldots, s_k) \in (\mathbb{Z}_{\geqslant 0})^k$. Let Γ be the comet-shaped quiver with g loops on the central vertex represented as:



Let $I = \{0\} \cup \{[i, j]\}_{1\leqslant i\leqslant k, 1\leqslant j\leqslant s_i}$ denote the set of vertices and let Ω be the set of arrows. For $\gamma \in \Omega$, we denote by $h(\gamma) \in I$ the head of γ and $t(\gamma) \in I$ the tail of γ. A *dimension vector* for Γ is a collection of non-negative integers $\mathbf{v} = \{v_i\}_{i\in I}$ and a *representation* φ of Γ of dimension $\mathbf{v}$ over a field $\mathbb{K}$ is a collection of $\mathbb{K}$-vector spaces $\{V_i\}_{i\in I}$ with $\dim V_i = v_i$ together with a collection of $\mathbb{K}$-linear maps $\{\varphi_\gamma : V_{t(\gamma)} \to V_{h(\gamma)}\}_{\gamma\in\Omega}$. We denote by $\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$ the $\mathbb{K}$-vector space of all representations of Γ over $\mathbb{K}$ of dimension vector $\mathbf{v}$. We also denote by $\mathrm{Rep}^*_{\mathbb{K}}(\Gamma, \mathbf{v})$ the subset of representations $\varphi \in \mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$ such that φ_γ is injective for all $\gamma \in \Omega$ such that $t(\gamma)$ is not the central vertex 0.

Assume from now that $\mathbb{K}$ is a finite field $\mathbb{F}_q$. We denote by $\mathrm{Rep}^{a.i}_{\mathbb{K}}(\Gamma, \mathbf{v})$ the set of absolutely indecomposable representations in $\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$. We also assume that $v_0 \neq 0$ so that $\mathrm{Rep}^{a.i}_{\mathbb{K}}(\Gamma, \mathbf{v}) \subset \mathrm{Rep}^*_{\mathbb{K}}(\Gamma, \mathbf{v})$. We may assume that $v_0 \geqslant v_{[i,1]} \geqslant \cdots \geqslant v_{i,s_i}$ for all i since otherwise $\mathrm{Rep}^{a.i}_{\mathbb{K}}(\Gamma, \mathbf{v}) = \emptyset$. For each i, take the strictly decreasing subsequence $v_0 > n_{i_1} > \cdots > n_{i_r}$ of $v_0 \geqslant v_{[i,1]} \geqslant \cdots \geqslant v_{i,s_i}$ of maximal length. This defines a partition $\mu^i := \mu^i_1 + \cdots + \mu^i_{r+1}$ of v_0 as follows: $\mu^i_1 = v_0 - n_{i_1}, \mu^i_2 = n_{i_1} - n_{i_2}, \ldots, \mu^i_{r+1} = n_{i_r}$. The dimension vector $\mathbf{v}$ defines thus a unique multipartition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$. The number $A_{\boldsymbol{\mu}}(q)$ of isomorphism classes in $\mathrm{Rep}^{a.i}_{\mathbb{K}}(\Gamma, \mathbf{v})$ depends only on $\boldsymbol{\mu}$ and not on $\mathbf{v}$.

Theorem 2.1. *(See [11].) For any* $\boldsymbol{\mu} \in \mathcal{P}^k$

$$A_{\boldsymbol{\mu}}(q) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}).$$

We know by a theorem of V. Kac that $A_{\boldsymbol{\mu}}(q) \in \mathbb{Z}[q]$, see [12]. It is also conjectured in [12] that the coefficients of $A_{\boldsymbol{\mu}}(q)$ are non-negative. Assuming Conjecture 1.1, Theorem 2.1 gives a cohomological interpretation of $A_{\boldsymbol{\mu}}(q)$; indeed, it implies that $A_{\boldsymbol{\mu}}(q)$ is the Poincaré polynomial of the pure part of the cohomology of $\mathcal{M}_{\boldsymbol{\mu}}$, thus implying Kac's conjecture for comet-shaped quivers. In particular, combining Conjecture 1.1 and Theorem 2.1 we obtain the conjectural equality of the middle Betti number of $\mathcal{M}_{\boldsymbol{\mu}}$ and $A_{\boldsymbol{\mu}}(1)$. These remarks can be compared to the fact that, when $\boldsymbol{\mu}$ is indivisible, $t^{d_\mu} A_\mu(t^2)$ is [4] the compactly supported Poincaré polynomial of $\mathcal{Q}_\mu$ and thus the middle Betti number of $\mathcal{Q}_\mu$ is $A_\mu(0)$.

Also, Theorems 1.6 and 2.1 imply that $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle = A_{\boldsymbol{\mu}}(q)$ (this does not use Conjecture 1.1). This gives an unexpected connection between the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ and that of comet-shaped quivers.

3. Connectedness of character varieties

The quiver variety $\mathcal{Q}_{\boldsymbol{\mu}}$ is known to be connected [2]. We use Theorem 1.3 to prove the following theorem.

Theorem 3.1. *(See [11].) The character variety* $\mathcal{M}_{\boldsymbol{\mu}}$*, if non-empty, is connected.*

Since the character variety $\mathcal{M}_{\boldsymbol{\mu}}$ is non-singular, the mixed Hodge numbers $h^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}})$ equal zero if $(i, j, k) \notin \{(i, j, k) \mid i \leqslant k, j \leqslant k, k \leqslant i + j\}$, see [5]. We thus have $h^0(\mathcal{M}_{\boldsymbol{\mu}}) = h^{0,0;0}(\mathcal{M}_{\boldsymbol{\mu}})$ and $h^{0,0;k}(\mathcal{M}_{\boldsymbol{\mu}}) = 0$ if $k > 0$. Hence by Corollary 1.4, we see that $h^0(\mathcal{M}_{\boldsymbol{\mu}})$ equals the constant term of the E-polynomial $E(\mathcal{M}_{\boldsymbol{\mu}}; q)$. To prove the theorem, we are thus reduced to prove that the coefficient of the lowest power $q^{\frac{1}{2}d_\mu}$ of q in $\mathbb{H}_{\boldsymbol{\mu}}(\sqrt{q}; 1/\sqrt{q})$ is 1. We use the following expansion [10, Lemma 5.1.5]:

$$\sum_{\boldsymbol{\mu}\in\mathcal{P}^k}\frac{q\mathbb{H}_{\boldsymbol{\mu}}(\sqrt{q},1/\sqrt{q})}{(q-1)^2}m_{\boldsymbol{\mu}}=\operatorname{Log}\left(\sum_{\lambda\in\mathcal{P}}\mathcal{H}_\lambda(\sqrt{q},1/\sqrt{q})\big(q^{-n(\lambda)}H_\lambda(q)\big)^k\prod_{i=1}^k s_\lambda(\mathbf{x}_i\mathbf{y})\right)$$

where $\mathbf{y}=\{1,1,q^2,\ldots\}$, $H_\lambda(q)$ is the hook polynomial and s_λ is the Schur symmetric function.

4. Relation with Hilbert schemes on $\mathbb{C}^*\times\mathbb{C}^*$

Put $X:=\mathbb{C}^*\times\mathbb{C}^*$ and denote by $X^{[n]}$ the Hilbert scheme of n points on X. We have [11]:

Theorem 4.1. *Assume that $g=1$ and $\boldsymbol{\mu}$ is the single partition $\mu=(n-1,1)$. Then $X^{[n]}$ and $\mathcal{M}_{\boldsymbol{\mu}}$ have the same mixed Hodge polynomial.*

The compactly supported mixed Hodge polynomial of $X^{[n]}$ is given by the following generating function [8]:

$$1+\sum_{n\geqslant 1}H_c\big(X^{[n]};q,t\big)T^n=\prod_{n\geqslant 1}\frac{(1+t^{2n+1}q^nT^n)^2}{(1-q^{n-1}t^{2n}T^n)(1-t^{2n+2}q^{n+1}T^n)}. \tag{4}$$

The identity (4) combined with the case $g=1$ and $\mu=(n-1,1)$ of our Conjecture 1.1 becomes the following purely combinatorial conjectural identity:

Conjecture 4.2.

$$1+\big(z^2-1\big)\big(1-w^2\big)\frac{\sum_\lambda\mathcal{H}_\lambda(z,w)\phi_\lambda(z^2,w^2)T^{|\lambda|}}{\sum_\lambda\mathcal{H}_\lambda(z,w)T^{|\lambda|}}=\prod_{n\geqslant 1}\frac{(1-zwT^n)^2}{(1-z^2T^n)(1-w^2T^n)}, \tag{5}$$

where $\phi_0:=0$ and if λ is a non-zero partition

$$\phi_\lambda(z,w):=\sum_{(i,j)\in\lambda}z^{j-1}w^{i-1},$$

where the sum runs over the boxes of λ.

Theorem 4.3. *Eq.* (5) *is true in the specialization* $(z,w)\mapsto(1/\sqrt{q},\sqrt{q})$.

This theorem is a consequence of (4), Theorems 1.3 and 4.1; in [11] we give an alternative purely combinatorial proof. Putting $q=e^u$ yields the following

Corollary 4.4.

$$1+\sum_{n\geqslant 1}\mathbb{H}_{(n-1,1)}\big(e^{u/2},e^{-u/2}\big)T^n=\frac{1}{u}\big(e^{u/2}-e^{-u/2}\big)\exp\left(2\sum_{k\geqslant 2}G_k(T)\frac{u^k}{k!}\right)$$

where $G_k(T)$, $k\geqslant 2$ are the standard Eisenstein series. In particular, the coefficient of any power of u of the left-hand side is in the ring of quasi-modular forms, *generated by the G_k, $k\geqslant 2$ over $\mathbb{Q}$.*

The fact that modular forms might be involved in this situation was pointed out in [7] and [1].

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