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Probability Theory

A new construction of the σ-finite measures associated with submartingales of class (Σ)

Une nouvelle construction des mesures σ-finies associées aux sous-martingales de classe (Σ)

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ABSTRACT

In Najnudel and Nikeghbali (2009) [7], we prove that for any submartingale $(X_t)_{t\geqslant 0}$ of class (Σ), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$, which satisfies some technical conditions, one can construct a σ-finite measure $\mathcal{Q}$ on $(\Omega, \mathcal{F})$, such that for all $t \geqslant 0$, and for all events $\Lambda_t \in \mathcal{F}_t$:

$$\mathcal{Q}[\Lambda_t, g \leqslant t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t]$$

where g is the last hitting time of zero of the process X. Some particular cases of this construction are related with Brownian penalisation or mathematical finance. In this Note, we give a simpler construction of $\mathcal{Q}$, and we show that an analog of this measure can also be defined for discrete-time submartingales.

RÉSUMÉ

Dans Najnudel et Nikeghbali (2009) [7], nous prouvons que pour toute sous-martingale $(X_t)_{t\geqslant 0}$ de classe (Σ), définie sur un espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$, satisfaisant certaines conditions techniques, on peut construire une mesure σ-finie $\mathcal{Q}$ sur $(\Omega, \mathcal{F})$, telle que pour tout $t \geqslant 0$, et pour tout événement $\Lambda_t \in \mathcal{F}_t$:

$$\mathcal{Q}[\Lambda_t, g \leqslant t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t]$$

où g est le dernier zéro de X. Certains cas particuliers de cette construction sont liés aux pénalisations browniennes ou aux mathématiques financières. Dans cette note, nous donnons une construction plus simple de $\mathcal{Q}$, et nous montrons qu'un analogue de cette mesure peut aussi être défini pour des sous-martingales à temps discret.

Version française abrégée

Les sous-martingales de classe (Σ), également appelées sous-martingales de Skorokhod dans [11], Chap. 3, ont été introduites par Yor (voir [13]) et certaines de leurs propriétés ont été étudiées par Nikeghbali dans [9]. Ces sous-martingales sont en particulier liées à certains problèmes de mathématiques financières (voir [4,1,3,11]) et aux pénalisations browniennes (voir [8]). La classe (Σ) est définie de la manière suivante :

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Définition 1.1. Soit $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ un espace de probabilité filtré. Une sous-martingale $(X_t)_{t\geqslant 0}$ est de classe (Σ), si pour tout $t \geqslant 0$, $X_t = N_t + A_t$, où $(N_t)_{t\geqslant 0}$ et $(A_t)_{t\geqslant 0}$ sont des processus $(\mathcal{F}_t)_{t\geqslant 0}$-adaptés satisfaisant les conditions suivantes :

- $(N_t)_{t\geqslant 0}$ est une martingale càdlàg ;
- $(A_t)_{t\geqslant 0}$ est un processus croissant, continu, tel que $A_0 = 0$;
- La mesure $(\mathrm{d}A_t)$ est portée par l'ensemble $\{t \geqslant 0, X_t = 0\}$.

Dans [7], nous prouvons qu'on peut associer une mesure σ-finie à toute sous-martingale de classe (Σ), définie sur un espace de probabilité filtré $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ satisfaisant une certaine condition technique, appelée propriété (NP) (le nom de cette propriété vient du fait qu'un espace qui la satisfait est l'augmentation naturelle d'un espace vérifiant une condition donnée par Parthasarathy : voir [5]). L'énoncé est le suivant :

Théorème 1.2. *Soit $(X_t)_{t\geqslant 0}$ une sous-martingale de classe (Σ) (ainsi, X_t est intégrable pour tout $t \geqslant 0$), définie sur un espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ qui satisfait la propriété (NP). En particulier, $(\mathcal{F}_t)_{t\geqslant 0}$ est continue à droite et $\mathcal{F}$ est la tribu engendrée par $\mathcal{F}_t$, $t \geqslant 0$. Dans ces conditions, il existe une unique mesure σ-finie $\mathcal{Q}$, définie sur $(\Omega, \mathcal{F}, \mathbb{P})$ et telle que si $g = \sup\{t \geqslant 0, X_t = 0\}$:*

- $\mathcal{Q}[g = \infty] = 0$;
- *Pour tout $t \geqslant 0$, et pour toute variable aléatoire F_t bornée, $\mathcal{F}_t$-mesurable,*

$$\mathcal{Q}[F_t \mathbb{1}_{g\leqslant t}] = \mathbb{E}_{\mathbb{P}}[F_t X_t]. \tag{1}$$

La propriété (NP) est définie précisément dans [5], et intervient de manière importante en raison de problèmes rencontrés lors d'extensions de familles compatibles de mesures de probabilités (ces problèmes sont également discutés par Bichteler dans [2]). Un exemple d'espace $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ satisfaisant la propriété (NP) peut être construit de la manière suivante :

- Ω est l'espace des fonctions continues de $\mathbb{R}_+$ dans $\mathbb{R}^d$, ou l'espace des fonctions càdlàg de $\mathbb{R}_+$ dans $\mathbb{R}^d$, pour un entier $d \geqslant 1$;
- on définit $(\mathcal{F}_t^0)_{t\geqslant 0}$ comme la filtration canonique associée à Ω, $\mathcal{F}^0$ comme la tribu engendrée par $(\mathcal{F}_t^0)_{t\geqslant 0}$, et on fixe une mesure de probabilité $\mathbb{P}^0$ sur $(\Omega, \mathcal{F}^0)$;
- on définit $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ comme la plus petite extension possible de $(\Omega, \mathcal{F}^0, \mathbb{P}^0, (\mathcal{F}_t^0)_{t\geqslant 0})$ telle que pour tout $t \geqslant 0$, $\mathcal{F}_0$ contient toutes les parties de Ω incluses dans un ensemble $A \in \mathcal{F}_t^0$ tel que $\mathbb{P}^0[A] = 0$.

Le Théorème 1.2 répond précisément à la question posée dans le Chapitre 3 de [11], concernant la représentation de type (1) de toute sous-martingale de classe (Σ). D'autre part, dans cette note, nous simplifions la preuve du Théorème 1.2 donnée dans [7]. Cependant, la preuve de [7] conserve son intérêt dans la mesure où elle est utilisée pour démontrer certaines propriétés importantes de la mesure $\mathcal{Q}$ (données dans [7] et [6]). Par ailleurs, nous prouvons une version du Théorème 1.2 valable pour des sous-martingales à temps discret. L'énoncé est le suivant :

Théorème 1.3. *Soit $(X_n)_{n\geqslant 0}$ une sous-martingale à temps discret (en particulier X_n est intégrable pour tout $n \geqslant 0$), définie sur un espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geqslant 0})$ qui satisfait une certaine propriété technique précisée plus bas. On suppose que pour tout $n \geqslant 0$, $X_n = N_n + A_n$, où $(N_n)_{n\geqslant 0}$ et $(A_n)_{n\geqslant 0}$ sont des processus satisfaisant les conditions suivantes :*

- *$(N_n)_{n\geqslant 0}$ est une martingale définie sur l'espace $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geqslant 0})$;*
- *$(A_n)_{n\geqslant 0}$ est un processus croissant, prévisible (i.e. A_n est $\mathcal{F}_{n-1}$-mesurable pour tout $n \geqslant 1$), tel que $A_0 = 0$;*
- *Pour tout $n \geqslant 0$, $(A_{n+1} - A_n)X_n = 0$, i.e. A ne peut croître qu'après les zéros de X.*

Dans ces conditions, il existe une unique mesure σ-finie $\mathcal{Q}$, définie sur $(\Omega, \mathcal{F}, \mathbb{P})$ et telle que si $g = \sup\{n \geqslant 0, X_n = 0\}$:

- $\mathcal{Q}[g = \infty] = 0$;
- Pour tout $n \geqslant 1$, et pour toute variable aléatoire F_n bornée, $\mathcal{F}_n$-mesurable,

$$\mathcal{Q}[F_n \mathbb{1}_{g<n}] = \mathbb{E}_{\mathbb{P}}[F_n X_n].$$

La preuve du Théorème 1.3 est exactement similaire à la nouvelle preuve du Théorème 1.2.

1. The continuous-time case

The submartingales of class (Σ), also called Skorokhod submartingales in [11], Chap. 3, were introduced by Yor in [13], and some of their main properties are studied by Nikeghbali in [9]. These submartingales are, in particular, related with some problems in mathematical finance (see [4,1,3,11]) and with Brownian penalisations (see [8]). The class (Σ) is defined as follows:

Definition 1.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. A nonnegative submartingale $(X_t)_{t\geqslant 0}$ is of class (Σ), if it can be decomposed as $X_t = N_t + A_t$ where $(N_t)_{t\geqslant 0}$ and $(A_t)_{t\geqslant 0}$ are $(\mathcal{F}_t)_{t\geqslant 0}$-adapted processes satisfying the following assumptions:

- $(N_t)_{t\geqslant 0}$ is a càdlàg martingale;
- $(A_t)_{t\geqslant 0}$ is a continuous increasing process, with $A_0 = 0$;
- The measure $(\mathrm{d}A_t)$ is carried by the set $\{t \geqslant 0, X_t = 0\}$.

In [7], we prove that one can associate a σ-finite measure to any submartingale of class (Σ), defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ satisfying a certain technical condition, called property (NP) (the name of this property comes from the fact that a space which satisfies it is the natural augmentation of a space enjoying a condition given by Parthasarathy: see [5]). The statement is the following:

Theorem 1.2. *Let $(X_t)_{t\geqslant 0}$ be a submartingale of the class (Σ) (in particular X_t is integrable for all $t \geqslant 0$), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ which satisfies the property (NP). In particular, $(\mathcal{F}_t)_{t\geqslant 0}$ is right-continuous and $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_t$, $t \geqslant 0$. Then, there exists a unique σ-finite measure $\mathcal{Q}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that for $g := \sup\{t \geqslant 0, X_t = 0\}$:*

- $\mathcal{Q}[g = \infty] = 0$;
- *For all $t \geqslant 0$, and for all $\mathcal{F}_t$-measurable, bounded random variables F_t,*

$$\mathcal{Q}[F_t \mathbb{1}_{g\leqslant t}] = \mathbb{E}_{\mathbb{P}}[F_t X_t]. \tag{2}$$

The property (NP) is defined precisely in [5], and plays an important role because of the problems encountered in the extension of compatible families of probability measures (these problems are also discussed by Bichteler in [2]). An example of a space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ satisfying the property (NP) can be constructed in the following way:

- Ω is the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, or the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, for an integer $d \geqslant 1$;
- One defines $(\mathcal{F}_t^0)_{t\geqslant 0}$ as the canonical filtration associated with Ω, $\mathcal{F}^0$ the σ-algebra generated by $(\mathcal{F}_t^0)_{t\geqslant 0}$, and one fixes a probability measure $\mathbb{P}^0$ on $(\Omega, \mathcal{F}^0)$;
- One defines $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ as the smallest possible extension of $(\Omega, \mathcal{F}^0, \mathbb{P}^0, (\mathcal{F}_t^0)_{t\geqslant 0})$ such that for all $t \geqslant 0$, $\mathcal{F}_0$ contains all the subsets of Ω included in a set $A \in \mathcal{F}_t^0$ such that $\mathbb{P}^0[A] = 0$.

Theorem 1.2 answers precisely the question posed in Chapter 3 of [11], concerning the representation of type (2) of any submartingale of class (Σ). Moreover, in this note, we simplify the proof of Theorem 1.2 given in [7]. However, the proof in [7] remains of interest, since it is used to establish some important properties of the measure $\mathcal{Q}$ (given in [7] and [6]). Moreover, we are able to prove an analog of Theorem 1.2 for a certain class of discrete-time submartingales. The precise statement is given in Section 2. The new proof of Theorem 1.2 and the proof of the discrete-time result are given in Section 3: these two proofs are very similar.

2. The discrete-time case

The discrete-time result is very similar to Theorem 1.2. However, one needs to define a discrete version of the property (NP), which we denote as (DP) (D standing for “discrete” and P for “Parthasarathy”). This can be done in the same way as for the continuous case. We first state the following definition:

Definition 2.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0})$ be a filtered measurable space, such that $\mathcal{F}$ is the σ-algebra generated by $\mathcal{F}_n$, $n \geqslant 0$: $\mathcal{F} = \bigvee_{n\geqslant 0} \mathcal{F}_n$. We shall say that the property (DP) holds if and only if $(\mathcal{F}_n)_{n\geqslant 0}$ enjoys the following conditions:

- For all $n \geqslant 0$, $\mathcal{F}_n$ is generated by a countable number of sets;
- For all $n \geqslant 0$, there exists a Polish space Ω_n, and a surjective map π_n from Ω to Ω_n, such that $\mathcal{F}_n$ is the σ-algebra of the inverse images, by π_n, of Borel sets in Ω_n, and such that for all $B \in \mathcal{F}_n$, $\omega \in \Omega$, $\pi_n(\omega) \in \pi_n(B)$ implies $\omega \in B$;
- If $(\omega_n)_{n\geqslant 0}$ is a sequence of elements of Ω, such that for all $N \geqslant 0$,

$$\bigcap_{n=0}^{N} A_n(\omega_n) \neq \emptyset,$$

where $A_n(\omega_n)$ is the intersection of the sets in $\mathcal{F}_n$ containing ω_n, then:

$$\bigcap_{n=0}^{\infty} A_n(\omega_n) \neq \emptyset.$$

A fundamental example of a space satisfying the property (DP) is obtained by taking $\Omega = X^{\mathbb{N}}$ where X is a Polish space, and $\mathcal{F} = \bigvee_{n\geqslant 0} \mathcal{F}_n$, where for $n \geqslant 0$, $\mathcal{F}_n$ is the σ-algebra of the inverse images, by the projection on the n first coordinates, of Borel sets in X^n. Another example can be constructed as follows:

- One defines Ω as the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, or as the space $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^d)$ of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ (for some $d \geqslant 1$);
- For $n \geqslant 0$, one defines $(\mathcal{F}_n)_{n\geqslant 0}$ as the natural filtration of the canonical process, and $\mathcal{F} = \bigvee_{n\geqslant 0} \mathcal{F}_n$.

The interest of property (DP) lies in the following result:

Proposition 2.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0})$ be a filtered measurable space satisfying the property (DP), and let, for $n \geqslant 0$, $(\mathbb{Q}_n)$ be a family of probability measures on $(\Omega, \mathcal{F}_n)$, such that for all $n \geqslant m \geqslant 0$, $\mathbb{Q}_m$ is the restriction of $\mathbb{Q}_n$ to $\mathcal{F}_m$. Then, there exists a unique measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that for all $n \geqslant 0$, its restriction to $\mathcal{F}_n$ is equal to $\mathbb{Q}_n$.*

Moreover, one can combine the property (DP) with the natural augmentation, defined by the following result:

Proposition 2.3. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ be a filtered probability space, and let $\mathcal{N}_0$ be the family of the subsets A of Ω which satisfy the following property: there exists $k \geqslant 0$ and $B \in \mathcal{F}_k$ such that $A \subset B$ and $\mathbb{P}[B] = 0$. One defines $\widetilde{\mathcal{F}}$ as the σ-algebra generated by $\mathcal{F}$ and $\mathcal{N}_0$, and for $n \geqslant 0$, $\widetilde{\mathcal{F}}_n$ as the σ-algebra generated by $\mathcal{F}_n$ and $\mathcal{N}_0$. Then, there exists a unique extension $\widetilde{\mathbb{P}}$ of $\mathbb{P}$, defined on $(\Omega, \widetilde{\mathcal{F}})$. The space $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_n)_{n\geqslant 0}, \widetilde{\mathbb{P}})$ is called the natural augmentation of the space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$.*

Note that in discrete case, the notion of a right-continuous filtration does not make sense. That is why the definition of the natural augmentation is slightly different from the definition in the continuous time case. The extension of measures is still available when one takes the natural augmentation of a space satisfying the property (DP). More precisely, one has the following result:

Proposition 2.4. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geqslant 0}, \mathbb{P})$ be the natural augmentation of a filtered probability space satisfying the property (DP). Then, for all coherent families of probability measures $(\mathbb{Q}_n)_{n\geqslant 0}$, such that $\mathbb{Q}_n$ is defined on $\mathcal{F}_n$, and is absolutely continuous with respect to the restriction of $\mathbb{P}$ to $\mathcal{F}_n$, there exists a unique probability measure $\mathbb{Q}$ on $\mathcal{F}$ which coincides with $\mathbb{Q}_n$ on $\mathcal{F}_n$ for all $n \geqslant 0$.*

The proofs of all these results are essentially given in [5] (the only change is the replacement of continuous filtrations by discrete filtrations). The method used in our proof of Proposition 2.2 comes from Stroock and Varadhan (see [12]). Note that the condition (DP) is not new and is essentially given by Parthasarathy in [10], p. 141. One can now state the discrete-time analog of Theorem 1.2.

Theorem 2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geqslant 0})$ be a filtered probability space satisfying the property (DP), or its natural augmentation. One considers a discrete-time submartingale $(X_n)_{n\geqslant 0}$ (in particular, X_n is integrable for all $n \geqslant 0$), defined on this space, and one supposes that for all $n \geqslant 0$, $X_n = N_n + A_n$, where the processes $(N_n)_{n\geqslant 0}$ and $(A_n)_{n\geqslant 0}$ satisfy the following conditions:*

- *$(N_n)_{n\geqslant 0}$ is a martingale defined on the space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geqslant 0})$;*
- *$(A_n)_{n\geqslant 0}$ is an increasing, predictable process (A_n is $\mathcal{F}_{n-1}$-measurable for all $n \geqslant 1$), such that $A_0 = 0$;*
- *For all $n \geqslant 0$, $(A_{n+1} - A_n)X_n = 0$, i.e. A can only increase after the zeros of X.*

Under these assumptions, there exists a unique σ-finite measure $\mathcal{Q}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and such that for $g := \sup\{n \geqslant 0, X_n = 0\}$:

- *$\mathcal{Q}[g = \infty] = 0$;*
- *For all $n \geqslant 1$, and for all bounded, $\mathcal{F}_n$-measurable random variables F_n,*

$$\mathcal{Q}[F_n \mathbb{1}_{g<n}] = \mathbb{E}_{\mathbb{P}}[F_n X_n].$$

The new proof of Theorem 1.2 and the proof of Theorem 2.5 are given in Section 3.

3. Proofs of Theorems 1.2 and 2.5

Let us first prove Theorem 1.2. For $t \geqslant 0$, we define d_t as the smallest hitting time of zero strictly after time t:

$$d_t := \inf\{s > t, X_s = 0\}.$$

By the début theorem, available for the spaces satisfying the property (NP) (as proven in [5]), d_t is a stopping time. One can prove that $(X_{s\wedge d_t})_{s\geqslant t}$ is a martingale with respect to the filtration $(\mathcal{F}_s)_{s\geqslant t}$ and the probability measure $\mathbb{P}$. Indeed, for all $s \geqslant t$,

$$X_{s\wedge d_t} = N_{s\wedge d_t} + A_{s\wedge d_t} = N_{s\wedge d_t} + A_t,$$

where, by the optional stopping theorem, $(N_{s\wedge d_t})_{s\geqslant 0}$ is a martingale, and A_t is $\mathcal{F}_s$-measurable and does not depend on s. Since $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ satisfies the property (NP), there exists a finite measure $\mathcal{Q}^{(t)}$ on $(\Omega, \mathcal{F})$ such that for all $s \geqslant t$, and for all events $\Lambda_s \in \mathcal{F}_s$:

$$\mathcal{Q}^{(t)}[\Lambda_s] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_s} X_{s\wedge d_t}].$$

Once $\mathcal{Q}^{(t)}$ is constructed, let us observe that under this measure, $g \leqslant t$ almost everywhere. Indeed, for all $s \geqslant t$:

$$\mathcal{Q}^{(t)}[d_t \leqslant s] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{d_t\leqslant s} X_{d_t}] = 0,$$

since for $d_t < \infty$, X_{d_t} vanishes by right-continuity of X. Hence, $\mathcal{Q}^{(t)}$-almost everywhere, $d_t = \infty$, or equivalently, $g \leqslant t$. Let us now prove that for $u \geqslant t \geqslant 0$, $\mathcal{Q}^{(t)}$ is the restriction of $\mathcal{Q}^{(u)}$ to the set $\{g \leqslant t\}$. Since $\mathcal{Q}^{(u)}$ almost-everywhere, X does not vanish strictly after time u, one has for all $s \geqslant u$, and for all events $\Lambda_s \in \mathcal{F}_s$:

$$\begin{aligned}\mathcal{Q}^{(u)}[g \leqslant t, \Lambda_s] &= \mathcal{Q}^{(u)}[d_t = \infty, \Lambda_s] = \mathcal{Q}^{(u)}[d_t > s, \Lambda_s] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{d_t>s,\Lambda_s} X_{d_u\wedge s}] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{d_t>s,\Lambda_s} X_s]\\ &= \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_s} X_{d_t\wedge s}] = \mathcal{Q}^{(t)}[\Lambda_s].\end{aligned}$$

Since $\mathcal{Q}^{(t)}$ is the restriction of $\mathcal{Q}^{(u)}$ to the set $\{g \leqslant t\}$, the family of measures $(\mathcal{Q}^{(t)})_{t\geqslant 0}$ is increasing. One can then define, for all events $\Lambda \in \mathcal{F}$:

$$\mathcal{Q}[\Lambda] = \lim_{t\to\infty} \mathcal{Q}^{(t)}[\Lambda],$$

where the limit is increasing. It is easy to check that $\mathcal{Q}$ is countably additive, hence, it is a σ-finite measure. Moreover, g is finite $\mathcal{Q}^{(t)}$-almost everywhere for all $t \geqslant 0$, and hence, $\mathcal{Q}$-almost everywhere. Now, for all $u \geqslant t \geqslant 0$, $\Lambda_t \in \mathcal{F}_t$:

$$\mathcal{Q}^{(u)}[g \leqslant t, \Lambda_t] = \mathcal{Q}^{(t)}[\Lambda_t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t],$$

and by taking $u \to \infty$:

$$\mathcal{Q}[g \leqslant t, \Lambda_t] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_t} X_t].$$

We have now proven the existence part of Theorem 1.2. The uniqueness part is very easily shown in [7]. Let us now sketch the proof of Theorem 2.5, which is similar to the proof of Theorem 1.2 given just above. For $n \geqslant 0$, let us define d_n as the first hitting time of zero strictly after time n: the process $(X_{p\wedge d_n})_{p\geqslant n+1}$ is a martingale with respect to the filtration $(\mathcal{F}_p)_{p\geqslant n+1}$. By Propositions 2.2 and 2.4, one deduces that there exists a finite measure $\mathcal{Q}^{(n)}$ such that its density with respect to $\mathbb{P}$, after restriction to $\mathcal{F}_p$, is equal to $X_{p\wedge d_n}$, for all $p \geqslant n+1$. Moreover, one has $g \leqslant n$, $\mathcal{Q}^{(n)}$-almost everywhere, and for $n \geqslant m$, $\mathcal{Q}^{(m)}$ is the restriction of $\mathcal{Q}^{(n)}$ to the set $g \leqslant m$. Hence, one can define $\mathcal{Q}$ as the increasing limit of $\mathcal{Q}^{(n)}$ when n goes to infinity. One obtains, for all $n \geqslant 1$, and for all events $\Lambda_n \in \mathcal{F}_n$:

$$\mathcal{Q}[g < n, \Lambda_n] = \mathcal{Q}^{(n-1)}[\Lambda_n] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\Lambda_n} X_n].$$

The uniqueness is proved as in the continuous-time case (see [7]).

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