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Number Theory

A property of the spectra of non-Pisot numbers

Une propriété du spectre des réels autres que les nombres de Pisot

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ABSTRACT

Let θ be a real number satisfying $1<\theta<2$, m a positive rational integer and $B_m(\theta)$ the set of polynomials with coefficients in $\{0,\pm 1,\dots,\pm m\}$, evaluated at θ. We prove that $B_m(\theta)$ is everywhere dense when $0\in B'_m(\theta)$, where $B'_m(\theta)$ is the derivative set of $B_m(\theta)$. We also show that if $B'_m(\theta)\cap[0,\ \frac{1}{\theta}\prod_{k\geqslant 0}(1-\frac{1}{\theta^{2^k}})]=\emptyset$, then $B_m(\theta)$ is discrete.

RÉSUMÉ

Soient θ un nombre réel satisfaisant $1<\theta<2$, m un entier rationnel positif et $B_m(\theta)$ l'ensemble des réels $P(\theta)$ pour P décrivant les polynômes à coefficients dans $\{0,\pm 1,\dots,\pm m\}$. On montre que $B_m(\theta)$ est partout dense lorsque 0 est un élément de l'ensemble dérivé $B'_m(\theta)$ de $B_m(\theta)$. On prouve également que si $B'_m(\theta)\cap[0,\ \frac{1}{\theta}\prod_{k\geqslant 0}(1-\frac{1}{\theta^{2^k}})]=\emptyset$, alors $B_m(\theta)$ est discret.

1. Introduction

We continue the investigation of the distribution in the real line $\mathbb{R}$ of the elements of the sets

$$B=B_m(\theta):=\{\varepsilon_0+\varepsilon_1\theta+\cdots+\varepsilon_n\theta^n,\ n\in\mathbb{N},\ \varepsilon_k\in\{-m,\dots,0,\dots,m\}\},$$

where θ is a real number satisfying $1<\theta<2$, and m runs through the set $\mathbb{N}$ of positive rational integers. The study of B has been initiated by Erdős, Joó and Komornik in [3], and followed by some authors (see for instance the references in [7]). A result of Bugeaud [2] asserts that all sets $B_m(\theta)$ are uniformly discrete if and only if θ is a Pisot number. A Pisot number is a real algebraic integer greater than 1, whose other conjugates over the field of the rationals $\mathbb{Q}$ are of modulus less than 1. Recall also that a subset X of $\mathbb{R}$ is uniformly discrete if the usual distance between two distinct elements of X is greater than a positive constant depending only on X; a uniformly discrete set is discrete, that is a set with no finite limit point. Notice also that $B_m(\theta)$ is uniformly discrete if and only if the quantity $\beta=\beta_m(\theta):=\inf\{b,\ b\in B_m(\theta),\ b>0\}$ satisfies $\beta_{2m}(\theta)>0$, or equivalently if and only if $0\notin B'_{2m}(\theta)$, where $B'=B'_m(\theta)$ is the set of limit points of $B_m(\theta)$. In [5] Erdős and Komornik considered the general case where the real number θ is not necessary less than 2. A corollary of one of their results asserts that if θ is not a Pisot number and $\theta\in\,]1,\frac{1+\sqrt{5}}{2}]$ (respectively, and $\theta\in\,]\frac{1+\sqrt{5}}{2},2[$), then $B_1(\theta)$ is not discrete (respectively, then $B_2(\theta)$ is not discrete and $\beta_3(\theta)=0$). A natural question arises immediately: *How can be distributed in* $\mathbb{R}$ *the elements of* B, *when* B *is not discrete*? The following result gives a partial answer to this problem:

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Theorem 1. *The set $B_m(\theta)$ is everywhere dense if and only if $\beta_m(\theta) = 0$.*

Combined with the above mentioned result of Erdős and Komornick, Theorem 1 yields:

Corollary. *If θ is not a Pisot number, $m \geqslant 2$ and $\theta \in]1, \frac{1+\sqrt{5}}{2}]$ (respectively, $m \geqslant 3$ and $\theta \in]\frac{1+\sqrt{5}}{2}, 2[$), then $B_m(\theta)$ is everywhere dense.*

Recall that the question whether there is a non-Pisot number, say again θ, satisfying $\beta_1(\theta) > 0$, has been cited in [4] and remains open. From the above we also see that a possible way to show that all sets $B_m(\theta)$ are everywhere dense when θ is not a Pisot number, is to prove the implication

$$B'_m(\theta) \neq \emptyset \quad \Rightarrow \quad \beta_m(\theta) = 0, \tag{1}$$

for $m = 1$ and $\theta \in]1, \frac{1+\sqrt{5}}{2}]$ (respectively, for $m \in \{1, 2\}$ and $\theta \in]\frac{1+\sqrt{5}}{2}, 2[$). In [1] Borwein and Hare have shown that $B_m(\theta)$ is discrete when $B_m(\theta) \cap [0, \frac{m}{\theta-1}]$ is finite, and the author [6] has proved that the bound $\frac{m}{\theta-1}$ may be replaced by the (non-optimal) constant $\frac{1}{\theta+1}$ without affecting the discreteness property of $B_m(\theta)$. The second aim of this note is to improve this last result:

Theorem 2. *The following propositions are equivalent*:

(i) *The set $B_m(\theta)$ is discrete.*
(ii) *The set $B'_m(\theta) \cap [0, \frac{1}{\theta} \prod_{k \geqslant 0}(1 - \frac{1}{\theta^{2^k}})]$ is empty.*

2. The proofs

Proof of Theorem 1. Trivially we have $\beta = 0$ when B is dense in $\mathbb{R}$. To make clear the proof of the converse we shall use the next result.

Lemma. *If $\beta = 0$, then the following properties hold*:

(i) *For any $\varepsilon > 0$ there exists $b \in B$ such that $\varepsilon < b \leqslant \theta\varepsilon$.*
(ii) *Each element of B is a limit point of B from both sides.*

Proof. (i) Since $\beta = 0$, there is $b_0 \in B$ such that $0 < b_0 < \varepsilon$. Let N be the greatest rational integer such that $\theta^N b_0 \leqslant \varepsilon$. Then, $\varepsilon < \theta^{N+1} b_0$, $\varepsilon < \theta^{N+1} b_0 \leqslant \theta\varepsilon$ and Lemma (i) follows, as $N + 1 \in \mathbb{N}$.

(ii) Since $\beta \in B'$ and $B = -B$, there is a decreasing sequence, say $(b_k)_{k\in\mathbb{N}}$, of distinct elements of B such that $\lim_{k\to\infty} b_k = 0$. Let $\varepsilon_0 + \varepsilon_1\theta + \cdots + \varepsilon_n\theta^n$, where $\varepsilon_i \in \{-m, \ldots, 0, \ldots, m\}$ and $n \in \mathbb{N}$, be a representation of an element $b \in B$. Then, $\theta^{n+1} b_k + b \in B$ and the sequence $(\theta^{n+1} b_k + b)_{k\in\mathbb{N}}$ is decreasing to b. Considering the sequence $(-\theta^{n+1} b_k + b)_{k\in\mathbb{N}}$, we see that b is a left-hand limit point of B. □

Let us return to the proof of Theorem 1, and assume on the contrary that $\beta = 0$ and B is not everywhere dense. Then, there exist positive numbers, say t_0 and δ, such that $[t_0, t_0 + \delta] \cap B = \emptyset$, as $B = -B$ and $0 \in B'$. Let $P = P(\delta) := \{t \in \mathbb{R},\ t > 0,\ [t, t + \delta] \cap B = \emptyset\}$. Then, $t_0 \in P$ and so $P \neq \emptyset$. We shall obtain a contradiction by considering the quantity $\alpha := \inf P$. First suppose $\alpha \in P$, and let

$$x \in \left]\max\left(\alpha - \frac{\delta}{2}, 0\right), \alpha\right[. \tag{2}$$

Then, $0 < x < \alpha$ and $[x, x+\delta] \cap B \neq \emptyset$. Let $b := \varepsilon_0 + \varepsilon_1\theta + \cdots + \varepsilon_n\theta^n$, where $\varepsilon_i \in \{-m, \ldots, 0, \ldots, m\}$ and $n \in \mathbb{N}$, be an element of $[x, x + \delta]$. Then, $b \in B$, and from the relations $x \leqslant b \leqslant x + \delta < \alpha + \delta$ and $[\alpha, \alpha + \delta] \cap B = \emptyset$, we have

$$b \in [x, \alpha[. \tag{3}$$

Since $\theta < 2$, Lemma (i) asserts that there is $b' \in B \cap]\frac{\alpha-x}{\theta^{n+1}}, 2\frac{\alpha-x}{\theta^{n+1}}[$, and by (2) we deduce that $\alpha - x < b'\theta^{n+1} < 2(\alpha - x) < \delta$. The last inequalities together with (3) yield $\alpha < b'\theta^{n+1} + b < \alpha + \delta$, and these relations lead to a contradiction, since $b'\theta^{n+1} + b \in B$ and $B \cap]\alpha, \alpha + \delta[= \emptyset$. Now, assume that $\alpha \notin P$. Then, $[\alpha, \alpha + \delta] \cap B \neq \emptyset$ and there is a decreasing sequence, say $(t_k)_{k\in\mathbb{N}}$, of distinct elements of P such that $\lim_{k\to\infty} t_k = \alpha$ and $t_k \leqslant \alpha + \delta$ for all $k \in \mathbb{N}$. Let $b \in [\alpha, \alpha + \delta] \cap B$. It follows by the relations $[t_k, t_k + \delta] \cap B = \emptyset$ and $\alpha < t_k \leqslant \alpha + \delta < t_k + \delta$ that

$$\alpha \leqslant b < t_k, \quad \forall k \in \mathbb{N}. \tag{4}$$

Letting k tend to infinity in (4), we obtain $\alpha = b$ and so $[\alpha, \alpha + \delta] \cap B = \{\alpha\}$; this last equality leads also to a contradiction because by Lemma (ii) the number α is a right-hand limit point of B and so the set $[\alpha, \alpha + \delta] \cap B$ contains certainly more than one element. □

Proof of the Corollary. Suppose that θ is not a Pisot number and $\theta \in]\frac{1+\sqrt{5}}{2}, 2[$ (respectively, and $\theta \in]1, \frac{1+\sqrt{5}}{2}]$). Then, $\beta_3(\theta) = 0$ (respectively, $B_1(\theta)$ is not discrete and so $\beta_2(\theta) = 0$ (this last equality has also been proved in [4])) and the result follows immediately by Theorem 1, since $B_m(\theta) \subset B_{m+1}(\theta)$ for all $m \in \mathbb{N}$. □

Proof of Theorem 2. Set $l = l_m(\theta) := \inf\{b',\ b' \in B' \cap [0, \infty[\}$ and $\ell := \frac{1}{\theta} \prod_{k \geqslant 0}(1 - \frac{1}{\theta^{2^k}})$. It is clear that $B' \cap [0, \ell] = \emptyset$ when B is discrete. Now, assume that B is not discrete. Then, Theorem 3 of [6] asserts that $l \leqslant \frac{1}{\theta+1}$. Let $(c_n)_{n \geqslant 0}$ be the sequence defined by $c_0 = 1$ and

$$c_n = \prod_{0 \leqslant k \leqslant n-1} \left(\theta^{2^k} - 1\right) \quad \text{for } n \in \mathbb{N}.$$

Then, $\frac{c_{n+1}}{\theta^{2^{n+1}}} = \frac{(\theta^{2^n} - 1)c_n}{\theta^{2^{n+1}}} < \frac{c_n}{\theta^{2^n} + 1} < \frac{c_n}{\theta^{2^n}}$,

$$\frac{c_n}{\theta^{2^n}} = \frac{1}{\theta} \prod_{0 \leqslant k \leqslant n-1} \left(1 - \frac{1}{\theta^{2^k}}\right)$$

and so $(\frac{c_n}{\theta^{2^n}})_{n \geqslant 0}$ is decreasing to ℓ. To show the inequality $l \leqslant \ell$, we shall prove that the propositions

$$B' \cap \left[\frac{c_n}{\theta^{2^n} + 1}, \frac{c_n}{\theta^{2^n}}\right] \neq \emptyset \quad \Rightarrow \quad B' \cap \left[0, \frac{c_n}{\theta^{2^n} + 1}\right] \neq \emptyset \tag{5}$$

and

$$B' \cap \left[\frac{c_{n+1}}{\theta^{2^{n+1}}}, \frac{c_n}{\theta^{2^n} + 1}\right] \neq \emptyset \quad \Rightarrow \quad B' \cap \left[0, \frac{c_{n+1}}{\theta^{2^{n+1}}}\right] \neq \emptyset \tag{6}$$

are true for all non-negative rational integers n. Indeed, if $l > \ell$, then there is $n_0 \in \mathbb{N}$ such that $\frac{c_{n_0}}{\theta^{2^{n_0}}} < l$. Let again n_0 be the smallest rational integer satisfying the last inequality. Then, from the relation $l \leqslant \frac{1}{\theta+1} < \frac{1}{\theta} = \frac{c_0}{\theta^{2^0}}$, we have $n_0 \geqslant 1$, $l \in]\frac{c_{n_0}}{\theta^{2^{n_0}}}, \frac{c_{n_0-1}}{\theta^{2^{n_0-1}}}] \cap B'$ and so by (5) (respectively, by (6)) we obtain a contradiction when $l > \frac{c_{n_0-1}}{\theta^{2^{n_0-1}}+1}$ (respectively, when $l \leqslant \frac{c_{n_0-1}}{\theta^{2^{n_0-1}}+1}$), since l is the smallest limit point of B. To show the relation (5) we consider the real function $f(x) = f_n(x) := -\theta^{2^n}x + c_n$, where $n \in \mathbb{N} \cup \{0\}$. It is clear that f is injective and continuous, and

$$f\left(\left[\frac{c_n}{\theta^{2^n} + 1}, \frac{c_n}{\theta^{2^n}}\right]\right) \subset \left[0, \frac{c_n}{\theta^{2^n} + 1}\right]. \tag{7}$$

Using the equality $c_{n+1} = c_n(\theta^{2^n} - 1)$, a simple induction shows that c_n is a monic polynomial in θ of degree $2^n - 1$ and with coefficients in $\{-1, 1\}$; thus $c_n = \theta^{2^n-1} \pm \theta^{2^n-2} \pm \cdots \pm 1$, $\forall n \in \mathbb{N}$, and so

$$\pm f(B) \subset B. \tag{8}$$

Hence, if $(a_k)_{k \in \mathbb{N}}$ is a sequence of distinct elements of B such that $\lim_{k \to \infty} a_k = a$ and $a \in [\frac{c_n}{\theta^{2^n}+1}, \frac{c_n}{\theta^{2^n}}]$, then the equality $\lim_{k \to \infty} f(a_k) = f(a)$ together with (7) and (8) give $f(a) \in B' \cap [0, \frac{c_n}{\theta^{2^n}+1}]$, and so the implication (5) is true. To prove the relation (6) notice first that we may suppose $\beta_1(\theta) > 0$, as $0 \in B'_m(\theta)$ for all $m \in \mathbb{N}$ when $\beta_1(\theta) = 0$. It follows by Remark 2 of [2] that θ is a root of a non-zero polynomial with coefficients in $\{-1, 0, 1\}$; thus θ is an algebraic integer and B is contained in the ring of integers of the field $\mathbb{Q}(\theta)$. Now, set $g := -f_{n+1}$. Then,

$$g\left(\left]\sum_{k=1}^{N} \frac{c_{n+1}}{\theta^{2^{n+1}k}}, \sum_{k=1}^{N+1} \frac{c_{n+1}}{\theta^{2^{n+1}k}}\right]\right) \subset \left]\sum_{k=1}^{N-1} \frac{c_{n+1}}{\theta^{2^{n+1}k}}, \sum_{k=1}^{N} \frac{c_{n+1}}{\theta^{2^{n+1}k}}\right],$$

where N runs through $\mathbb{N}$ (by convention $\sum_{k=1}^{0} \frac{c_{n+1}}{\theta^{2^{n+1}k}} := 0$). Notice also that for each $x \in]\frac{c_{n+1}}{\theta^{2^{n+1}}}, \frac{c_n}{\theta^{2^n}+1}[$ there exists a unique positive rational integer, say $N(x)$, such that

$$\sum_{k=1}^{N(x)} \frac{c_{n+1}}{\theta^{2^{n+1}k}} < x \leqslant \sum_{k=1}^{N(x)+1} \frac{c_{n+1}}{\theta^{2^{n+1}k}},$$

because $\sum_{k\geqslant 1}\frac{c_{n+1}}{\theta^{2^{n+1}k}}=\frac{c_{n+1}}{\theta^{2^{n+1}}-1}=\frac{c_{n+1}}{(\theta^{2^n}-1)(\theta^{2^n}+1)}=\frac{c_n}{\theta^{2^n}+1}$. Consequently, if $t\in\,]\frac{c_{n+1}}{\theta^{2^{n+1}}},\frac{c_n}{\theta^{2^n}+1}[$ and $(t_k)_{k\in\mathbb{N}}$ is a sequence of distinct elements of B satisfying $\lim_{k\to\infty}t_k=t$, then $\lim_{k\to\infty}g(t_k)=g(t)$, and so $g(t)\in B'$ (recall that g is injective and continuous, and by (8) we have $g(B)\subset B$). Hence, $g(t)\in B'\cap\,]\sum_{k=1}^{N(t)-1}\frac{c_{n+1}}{\theta^{2^{n+1}k}},\sum_{k=1}^{N(t)}\frac{c_{n+1}}{\theta^{2^{n+1}k}}]$. Iterating the map g we deduce that $g^{(N(t))}(t)\in B'\cap\,]0,\frac{c_{n+1}}{\theta^{2^{n+1}}}]$, and so the implication (6) is true when $B'\cap[\frac{c_{n+1}}{\theta^{2^{n+1}}},\frac{c_n}{\theta^{2^n}+1}]\neq\{\frac{c_n}{\theta^{2^n}+1}\}$. Finally, let us consider the case where $B'\cap[\frac{c_{n+1}}{\theta^{2^{n+1}}},\frac{c_n}{\theta^{2^n}+1}]$ is reduced to the singleton $\{\frac{c_n}{\theta^{2^n}+1}\}$. Notice first by (7) and (8) that $\frac{c_n}{\theta^{2^n}+1}$ is a left-hand limit point of B. Furthermore, if $(s_k)_{k\in\mathbb{N}}$ is a sequence of distinct elements of $B\cap\,]\frac{c_{n+1}}{\theta^{2^{n+1}}},\frac{c_n}{\theta^{2^n}+1}[$ such that $\lim_{k\to\infty}s_k=\frac{c_n}{\theta^{2^n}+1}$, then by the above for each $k\in\mathbb{N}$ there is $N_k:=N(s_k)\in\mathbb{N}$ such that

$$g^{(N_k)}(s_k)\in\left]0,\frac{c_{n+1}}{\theta^{2^{n+1}}}\right]\cap B;$$

thus if the set $E:=\{g^{(N_k)}(s_k),\ k\in\mathbb{N}\}$ is not finite, then B has a limit point which belongs to the interval $[0,\frac{c_{n+1}}{\theta^{2^{n+1}}}]$, and so (6) is true. Now, assume on the contrary that E is finite and let $b\in B\cap\,]0,\frac{c_{n+1}}{\theta^{2^{n+1}}}]$ such that $b=g^{(N_k)}(s_k)$ for infinitely many k. Writing $g^{(N_k)}(s_k)$ explicitly, the last equality gives

$$b=\gamma^{N_k}s_k-\gamma^{(N_k-1)}c_{n+1}-\cdots-\gamma c_{n+1}-c_{n+1},\tag{9}$$

where $\gamma:=\theta^{2^{n+1}}$. Hence, if σ is an embedding of $\mathbb{Q}(\theta)$ into the complex field $\mathbb{C}$, sending θ to a conjugate over $\mathbb{Q}$ of modulus at least 1, then (9) implies

$$\sigma(b)=\sigma(\gamma)^{N_k}\sigma(s_k)-\sigma(c_{n+1})\frac{\sigma(\gamma)^{N_k}-1}{\sigma(\gamma)-1},$$

$$\sigma(s_k)=\frac{\sigma(b)}{\sigma(\gamma)^{N_k}}+\sigma(c_{n+1})\frac{1-\frac{1}{\sigma(\gamma)^{N_k}}}{\sigma(\gamma)-1}$$

$(\sigma(\gamma)\notin\{0,1\}$ because $\theta>1)$, and so

$$\left|\sigma(s_k)\right|\leqslant\left|\sigma(b)\right|+2\left|\frac{\sigma(c_{n+1})}{\sigma(\gamma)-1}\right|.\tag{10}$$

Notice also that if $\varepsilon_0+\varepsilon_1\theta+\cdots+\varepsilon_d\theta^d$ is a representation in B of some s_k, and if τ is an embedding of $\mathbb{Q}(\theta)$ into $\mathbb{C}$ sending θ to a conjugate of modulus less than 1, then

$$\left|\tau(s_k)\right|\leqslant m\sum_{k=0}^{d}\left|\tau(\theta)\right|^k<\frac{m}{1-|\tau(\theta)|}.\tag{11}$$

It follows by (10) and (11) that the conjugates of the integer s_k of the field $\mathbb{Q}(\theta)$ are bounded; thus s_k takes at most a finite number of values and this is absurd, as $\{s_k,\ k\in\mathbb{N}\}$ is not finite. □

Remark 1. By the same method as in the proof of Theorem 2, we easily obtain $l\notin\,]\frac{P_n}{\theta^n+1},\ \frac{P_n}{\theta^n-1}[$, where $n\in\mathbb{N}$, $P_n=\varepsilon_{n-1}\theta^{n-1}+\varepsilon_{n-2}\theta^{n-2}+\cdots+\varepsilon_0$ and $\varepsilon_i\in\{-m,\dots,0,\dots,m\}$. I am not able to prove (or disprove) the inclusion: $]0,1[\,\subset\bigcup_{n\in\mathbb{N}}]\frac{P_n}{\theta^n+1},\ \frac{P_n}{\theta^n-1}[$, which implies (1).

Remark 2. With the notation of the proof of Theorem 2, suppose $\beta\neq 0$. Then, each finite sum, say s, of the form $\frac{\varepsilon_1}{\theta}+\cdots+\frac{\varepsilon_N}{\theta^N}$, where $\varepsilon_i\in\{-m,\dots,0,\dots,m\}$ and $N\in\mathbb{N}$, does not belong to B'. Indeed, if $(b_k)_{k\in\mathbb{N}}$ is a sequence of distinct elements of B such that $\lim_{k\to\infty}b_k=s$, then $\theta^Nb_k-(\varepsilon_1\theta^{N-1}+\cdots+\varepsilon_N)\in B$ and $\lim_{k\to\infty}\theta^Nb_k-\varepsilon_1\theta^{N-1}-\cdots-\varepsilon_N=0$. In particular for $m=1$ we have $L_1(\theta):=\sup\{b',b'\in B_1'(\theta)\cap[0,1]\}<1$, since by Remark 2 of [2] there are $N\in\mathbb{N}$ and $\varepsilon_i\in\{-1,0,1\}$ such that $1=\frac{\varepsilon_1}{\theta}+\cdots+\frac{\varepsilon_N}{\theta^N}$; thus $0<\beta_1(\theta)\leqslant l_1(\theta)\leqslant\ell<\frac{1}{\theta}<L_1(\theta)<1$.

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