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Probability Theory

The survival probability of a critical branching process in a Markovian random environment

La probabilité de survie d'un processus de branchement critique en environnement aléatoire Markovien

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ABSTRACT

In this Note, we first prove a local limit theorem for a semi-Markov chain and then apply it to study the asymptotic behavior of the survival probability of a critical branching process in Markovian random environment.

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RÉSUMÉ

Dans cette Note, nous montrons d'abord un théorème de la limite locale pour une chaîne semi-Markovienne. Nous appliquons ensuite ce résultat pour étudier le comportement asymptotique de la probabilité de survie d'un processus de branchement critique dans un milieu aléatoire Markovien.

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1. Introduction and main results

The study of branching processes in Markovian random environment has been developed by several authors, in particular by K.B. Athreya and S. Karlin [1]. However, the asymptotic behavior of the survival probability of such a process is not yet known. In this Note we handle this problem in the case of a critical branching process.

Consider the following model: $X = (X_n)_{n\geqslant 0}$ is an irreducible and aperiodic Markov chain on a finite space E with transition matrix P. The chain X has a unique invariant probability ν. We denote by G the set of generating functions of probability measures on $\mathbb{N}$, equipped with the topology of simple convergence on $[0,1]$. $\mathcal{B}(G)$ is the Borel σ-algebra on G. In addition, we define a Markov chain $(M_n)_{n\geqslant 0} = (g_n, X_n)_{n\geqslant 0}$ with values in $G \times E$ and with transition probability Q defined by

$$Q\big\{(g,i), \big(A \times \{j\}\big)\big\} = P(i,j)\overline{F}(i,j,A), \quad \text{for } (g,i) \in G \times E,\ A \in \mathcal{B}(G),$$

where $\overline{F}$ is a transition probability from $E \times E$ in the set of probabilities on G. The Markov chain $(M_n)_{n\geqslant 0}$ is called the *environment process*. Let $\Omega = (G \times E)^{\mathbb{N}}$ and $\mathcal{F} = \bigotimes^{\mathbb{N}}(\mathcal{B}(G) \otimes \mathcal{P}(E))$. We denote by $\mathbb{P}_{(g,i)}$ the unique probability on $(\Omega, \mathcal{F})$, such that for any $(g,i) \in G \times E$, any $n \geqslant 1$ and any bounded measurable function $f : (G \times E)^n \to \mathbb{R}$, we get

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$$\int_\Omega f\big(M_0(\omega), M_1(\omega), \ldots, M_n(\omega)\big)\mathbb{P}_{(g,i)}(\mathrm{d}\omega)$$
$$= \sum_{(j_1,j_2,\ldots,j_n)\in E^n} P(i,j_1)\cdots P(j_{n-1},j_n)\int_{G^n} f\big((g,i),(g_1,j_1),\ldots,(g_n,j_n)\big)\overline{F}(i,j_1,\mathrm{d}g_1)\cdots\overline{F}(j_{n-1},j_n,\mathrm{d}g_n).$$

To simplify the notations, $\mathbb{P}_{(Id,i)}$ will be denoted by $\mathbb{P}_i$ and $\mathbb{E}_i$ is its corresponding expectation.

Given $(M_n)_{n\geqslant 0}$, we define now the branching process $(Z_n)_{n\geqslant 0}$ such that $Z_0 = 1$ and the generating function of Z_n is

$$g_0 \circ g_1 \circ \cdots \circ g_{n-1}(s) := G_n(s), \quad 0 \leqslant s < 1.$$

Therefore, given $(M_n)_{n\geqslant 0}$, the survival probability of the branching process $(Z_n)_{n\geqslant 0}$ at time n is

$$1 - G_n(0) := q_n.$$

Due to the Markov property of the probability $\mathbb{P}_i$, we have for $(i,j) \in E \times E$ and $n \geqslant 1$,

$$\mathbb{P}_i(Z_n > 0, X_n = j) = \mathbb{E}_i\big(q_n P(X_{n-1}, j)\big).$$

Let's consider $h : G \to \overline{\mathbb{R}}_+$, $g \mapsto h(g) := g'(1)$. The image of the probability $\overline{F}(i,j,\mathrm{d}x)$ by the map h is denoted by $F(i,j,\mathrm{d}x)$. We assume in this paper the following hypotheses (H):

(H1) there exist $\alpha > 0$, such that for all $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Re}\,\lambda| \leqslant \alpha$, we have

$$\sup_{(i,j)\in E\times E} \big|\widehat{F}(i,j,\lambda)\big| < +\infty, \quad \text{where } \widehat{F}(i,j,\lambda) = \int_{\mathbb{R}} e^{\lambda t} F(i,j,\mathrm{d}t);$$

(H2) there exist $n_1 \geqslant 1$ and $(i_0, j_0) \in E \times E$, such that the measure $\mathbb{P}_{i_0}(X_{n_1} = j, S_{n_1} \in \mathrm{d}x)$ has an absolutely continuous component with respect to the Lebesgue measure on $\mathbb{R}$;

(H3) $\sum_{(i,j)\in E\times E} \nu(i)P(i,j)\int_{\mathbb{R}} tF(i,j,\mathrm{d}t) = 0$.

By [1], the hypothesis (H3) implies

$$\mathbb{P}_\nu(Z_n = 0) \to 1, \quad \text{as } n \to +\infty.$$

Such a branching process $(Z_n)_{n\geqslant 0}$ is called *critical*.

Let us introduce some notations: set

$$\eta_{k,n} := f_k\big(g_{k+1,n}(0)\big),$$

where

$$f_k(s) := \frac{1}{1-g_k(s)} - \frac{1}{g_k'(1)(1-s)}, \quad \text{for } 0 \leqslant s < 1 \quad \text{and} \quad g_{k,n} := g_k \circ g_{k+1} \circ \cdots \circ g_{n-1}, \quad \text{for } 0 \leqslant k \leqslant n-1;$$

$$S_n := Y_0 + Y_1 + \cdots + Y_{n-1}, \quad \text{for } n \geqslant 1, \quad \text{with } S_0 := 0 \quad \text{and} \quad Y_k := \ln g_k'(1), \quad \text{for } k \geqslant 0.$$

Then we can obtain the following formula [4]:

$$q_n^{-1} = \exp(-S_n) + \sum_{k=0}^{n-1} \eta_{k,n}\exp(-S_k). \tag{1}$$

Theorem 1.1. *Under hypotheses* (H)*, for any* $(i,j) \in E \times E$*, there exists a constant* $\beta_{i,j} > 0$*, such that*

$$\lim_{n\to+\infty} \sqrt{n}\,\mathbb{P}_i(Z_n > 0, X_n = j) = \beta_{i,j}. \tag{2}$$

For $n \geqslant 0$, we set

$$m_n = \min\{S_0, S_1, \ldots, S_n\}.$$

The proof of Theorem 1.1 is based on the following local limit theorem:

Theorem 1.2. *Under hypotheses* (H), *for any* $(i, j) \in E \times E$ *and* $x \geqslant 0$, *we get*

$$\lim_{n \to +\infty} \sqrt{n}\mathbb{P}_i(m_n \geqslant -x, X_n = j) = h_{i,j}(x) > 0, \tag{3}$$

where $h_{i,j}$ *is an increasing harmonic function for* $(S_n, X_n)_{n \geqslant 0}$ *on* $\mathbb{R}_+ \times E$.

Furthermore, there exists a constant $\sigma^2 > 0$, *such that*

$$h_{i,j}(x) \sim \sqrt{\frac{2}{\sigma^2}} \nu(j)x, \quad x \to +\infty. \tag{4}$$

J. Geiger and G. Kersting [4], Y. Guivarc'h, E. Le Page and Q. Liu [5] proved an analog of Theorem 1.1 in the case of i.i.d. environment under weaker moment assumptions and without any hypotheses of absolute continuity of type (H2).

In the case when E contains one single point, Theorem 1.2 extends the local limit theorem for the minimum of a random walk on $\mathbb{R}$ (see also [2]). Theorem 1.2 improves the results of E.L. Presman [6], especially we prove that for any $(i, j) \in E \times E$, the limit function $h_{i,j}$ defined in (3) does not vanish and we specify its asymptotic behavior as $x \to +\infty$.

2. Sketch of proofs

To prove Theorem 1.2, we make use of a factorization method due to E.L. Presman [6]. We denote by $L_\infty(E)$ the space of bounded function on E, equipped with the uniform norm. We define matrices $\mathcal{P}B_z(\lambda)$, $\mathcal{Q}C_z(\lambda)$, and Fourier–Laplace operators $P(\lambda)$ on $L_\infty(E)$ as follows:

$$\mathcal{P}B_z(\lambda) = \left[\sum_{n=1}^{+\infty} z^n \mathbb{E}_i\big(e^{\lambda S_n}; S_1 > S_n, S_2 > S_n, \ldots, S_{n-1} > S_n, S_n < 0; X_n = j\big) \right]_{i,j},$$

$$\mathcal{Q}C_z(\lambda) = \left[\sum_{n=1}^{+\infty} z^n \mathbb{E}_i\big(e^{\lambda S_n}; S_1 \geqslant 0, S_2 \geqslant 0, \ldots, S_{n-1} \geqslant 0, S_n \geqslant 0; X_n = j\big) \right]_{i,j},$$

$$P(\lambda)\varphi(i) = \sum_{j \in E} P(i, j)\varphi(j)\widehat{F}(i, j, \lambda) = \sum_{j \in E} P(i, j)\varphi(j) \int_{\mathbb{R}} e^{\lambda t} F(i, j, \mathrm{d}t),$$

for $\varphi \in L_\infty(E)$, $i \in E$ and $|\operatorname{Re}\lambda| < \alpha$.

We first prove that the matrix $H(z, \lambda) := \sqrt{1-z}\,[\sum_{n=0}^{+\infty} z^n \mathbb{E}_i(e^{\lambda m_n}; X_n = j)]$ can be factorized as follows: for $|z| < 1$, $\operatorname{Re}\lambda = 0$,

$$H(z, \lambda) = \big[I + \mathcal{P}B_z(\lambda)\big]\sqrt{1-z}\big[I + \mathcal{Q}C_z(0)\big].$$

In addition, we have the following identity, which is analogous to the well-known Wiener–Hopf factorization [3],

$$\big(I - zP(\lambda)\big)^{-1} = \big[I + \mathcal{P}B_z(\lambda)\big]\big[I + \mathcal{Q}C_z(\lambda)\big], \quad |z| < 1,\ \operatorname{Re}\lambda = 0. \tag{5}$$

Then using E.L. Presman's factorization theory [6] and especially analytical properties of such factorization, we can prove that

(1) for $\operatorname{Re}\lambda > 0$, the function $[I + \mathcal{P}B_z(\lambda)]$ is analytic with respect to z in $D_{\rho,\theta} = \{z;\ z \neq 1,\ |\arg(z-1)| \geqslant \theta > 0,\ |z| < \rho\}$, $\rho > 1$, $0 < \theta < \pi/2$ and admits an analytical extension to the boundary of $D_{\rho,\theta}$;
(2) as $\lambda \to 0$, the limit of $\lambda[I + \mathcal{P}B_1(\lambda)]$ exists;
(3) $[I + \mathcal{Q}C_z(0)]$ is analytic with respect to z in $D_{\rho,\theta}$. Furthermore, $\sqrt{1-z}[I + \mathcal{Q}C_z(0)]$ is bounded in $D_{\rho,\theta}$ and admits a limit as $z \to 1$.

So for any $(i, j) \in E \times E$, the limit of $[H(z, \lambda)]_{i,j}$ as $z \to 1$ exists and is denoted by $[H(\lambda)]_{i,j}$, from which, leads to (3), using complex analysis argument. Moreover, we get

$$\lim_{\lambda \to 0^+} \lambda\big[H(\lambda)\big]_{i,j} = \sqrt{\frac{2}{\sigma^2}} \nu(j) > 0,$$

where σ^2 is a positive constant. Using Tauberian theorem [3], we obtain (4).

The proof of Theorem 1.1 is similar to the one in [4]. An important step is to check, using Theorem 1.2 and the formula (1), that for any $\varrho > 1$, $x \geqslant 0$ and $i \in E$, we have

$$\limsup_{m \to +\infty} \limsup_{n \to +\infty} \sqrt{n}\mathbb{P}_i(Z_m > 0, Z_n = 0, m_{\varrho n} \geqslant -x) = 0. \tag{6}$$

3. More information on the proof of Theorem 1.1

For every $x \geqslant 0$ and every $(i, j) \in E \times E$, let denote by $\widehat{\mathbb{E}}_{(i,j,x)}$ the expectation corresponding to the unique probability $\widehat{\mathbb{P}}_{(i,j,x)}$ on $(\Omega, \mathcal{F})$ such that for every integer $n \geqslant 1$ and every measurable, bounded function $f : (G \times E)^n \to \mathbb{R}$, we have

$$\int_{\Omega} f\big(M_1(\omega), \ldots, M_n(\omega)\big)\widehat{\mathbb{P}}_{(i,j,x)}(\mathrm{d}\omega) = \frac{1}{h_{i,j}(x)} \int_{\Omega} f\big(M_1(\omega), \ldots, M_n(\omega)\big) h_{X_n(\omega),j}\big(x + S_n(\omega)\big)\mathbb{P}_i(\mathrm{d}\omega).$$

Set $q_\infty = \lim_{n\to+\infty} q_n$. Using the equality (6) and Theorem 1.2, we can establish that

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}_i(Z_n > 0, X_n = j) = \lim_{x\to+\infty} h_{i,j}(x)\widehat{\mathbb{E}}_{(i,j,x)}(q_\infty).$$

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