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On the existence and conditional energetic stability of solitary water waves with weak surface tension

Sur l'existence et la stabilité des ondes solitaires en présence d'une faible tension superficielle

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A R T I C L E I N F O

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A B S T R A C T

An existence and stability theory for solitary water waves with weak surface tension has recently been given by Buffoni (2005, 2009) [2,3]. The theory, which is variational in nature, relies upon the assumption that the infimum of the variational functional is strictly subhomogeneous with respect to a small parameter. In this Note we rigorously establish the relevant strict-subhomogeneity property and thus complete Buffoni's theory.

R É S U M É

Récemment, Buffoni (2005, 2009) [2,3] a développé une théorie d'existence et de stabilité des ondes solitaires de surface dans le cas d'une faible tension de surface. Cette théorie, qui est de nature variationnelle, repose sur l'hypothèse que l'infimum d'une certaine fonctionnelle variationnelle est strictement sous-homogène par rapport à un petit paramètre. Dans cette Note, on démontre cette propriété de sous-homogénéité stricte complétant ansi la théorie de Buffoni.

Version française abrégée

Les points critiques de la fonctionnelle $\mathcal{J}_\mu$ définie par (1) appartenant à l'ensemble $U \setminus \{0\} = \{u \in H^2(\mathbb{R})\colon\ 0 < \|u\|_2 < R\}$ correspondent aux ondes solitaires de petite amplitude à la surface libre d'un fluide dans un domaine $\{(\xi, y)\colon\ y \in (-1, \eta(\xi))\}$ avec un nombre de Bond $\beta > 0$, où $\eta(\xi) = u(x)$, $\xi = x + \int_0^x (Nu)(s)\,ds$. Le théorème suivant a été démontré par Buffoni [1,2] (une définition précise de la notion de 'stabilité énergétique conditionnelle' est donnée dans ces articles).

Théorème 0.1.

(i) *Supposons que toute suite minimisante $\{u_n\}$ de $\mathcal{J}_\mu$ sur $U \setminus \{0\}$ telle que $\sup_n \|u_n\|_2 < R$ converge (à une sous-suite et à une translation près) faiblement dans $H^2(\mathbb{R})$ et fortement dans $H^s(\mathbb{R})$, $s \in [0, 2)$. Alors l'ensemble des minimiseurs de $\mathcal{J}_\mu$ sur $U \setminus \{0\}$ définit une famille d'ondes solitaires qui sont conditionnellement énergétiquement stables.*

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(ii) *Soit* $c_\mu = \inf_{u \in U \setminus \{0\}} \mathcal{J}_\mu(u)$. *Si la propriété de sous-homogénéité stricte* $c_{a\mu} < ac_\mu$ *est vraie pour tout* $a > 1$, *alors le critère de convergence au point* (i) *est vérifié.*

Buffoni [1] a montré que la propriété de sous-homogénéité stricte est vérifiée si $\beta > 1/3$. Dans cette note on montre qu'elle est aussi vérifiée si $\beta < 1/3$.

On commence par l'étude des fonctions $u \in U \setminus \{0\}$ telles que $\|u\|_2^2 \leqslant c\mu$, $\mathcal{J}_\mu(u) < 2\mu$, si $\|\mathcal{J}'_\mu(u)\|_0 \leqslant \mu^N$ pour un certain $N \geqslant 3$, où $\mathcal{J}'_\mu$ est le gradient de $\mathcal{J}_\mu$ dans $L^2(\mathbb{R})$. Une fonction de ce type est l'approximation d'un minimiseur de $\mathcal{J}_\mu$ sur $U \setminus \{0\}$, et des résultats précédents basés sur une réduction à une variété centrale (Iooss et Kirchgässner [5], Buffoni et Groves [4]) indiquent que le spectre de u est concentré autour de ω (le nombre réel positif minimisant $(1+\beta k^2)/(|k|\coth|k|)$). Soient $\delta \in (0, \omega/3)$ et χ la fonction caractéristique de l'ensemble $[-\omega-\delta, -\omega+\delta] \cup [\omega-\delta, \omega+\delta]$. On écrit $u = u_1 - \mathcal{G}(u_1) + u_3$, où u_1, $\mathcal{G}(u_1)$ et u_3 sont définis respectivement par (4), (6) et (5). Alors une série d'estimations précises montre que les inégalités $|||u_1|||_\alpha^2 \leqslant c\mu$, $\|u_3\|_2^2 \leqslant c\mu^{3+2\alpha}$ et $\|\mathcal{G}(u_1)\|_2^2 \leqslant c\mu^{2+\alpha}$ sont vérifiées pour tout $\alpha < 1$, où

$$|||u_1|||_\alpha^2 := \int_0^\infty \left(1+\mu^{-4\alpha}|k-\omega|^4\right)\left|\hat{u}_1(k)\right|^2 \mathrm{d}k + \int_{-\infty}^0 \left(1+\mu^{-4\alpha}|k+\omega|^4\right)\left|\hat{u}_1(k)\right|^2 \mathrm{d}k.$$

Sous l'hypothèse plus forte $\mathcal{J}_\mu(u) < 2\mu - c\mu^3$, on déduit en utilisant ces estimations que $\mathcal{M}(u) = \mathcal{K}_3(u) + \mathcal{K}_4(u) + o(\mu)$, $\mathcal{K}_3(u) = A_1(\omega)\int_\mathbb{R} u_1^4\,\mathrm{d}x + o(\mu^3)$ et $\mathcal{K}_4(u) = A_2(\omega)\int_\mathbb{R} u_1^4\,\mathrm{d}x + o(\mu^3)$, où $\mathcal{M}(u)$, $\mathcal{K}_3(u)$, $\mathcal{K}_4(u)$ sont respectivement les parties super-quadratique, quadratique et cubique de $\mathcal{K}(u)$, et $A_1(\omega)$, $A_2(\omega)$ sont données par des formules explicites.

Finalement, on applique nos résultats à une suite minimisante $\{u_n\}$ de $\mathcal{J}_\mu$ sur $U \setminus \{0\}$ telle que $\|\mathcal{J}'(\tilde{u}_n)\|_0 \to 0$ lorsque $n \to +\infty$, dont l'existence a été démontrée par Buffoni [2]. On montre que la fonction $a \mapsto a^{-5/2}\mathcal{M}(a\tilde{u}_n)$, $a \in [1,2]$, est strictement décroissante et négative, donc en particulier $\mathcal{M}(a^{1/2}\tilde{u}_n) \leqslant a^{5/4}\mathcal{M}(\tilde{u}_n)$, pour tout $a \in (1,4]$. On déduit ensuite l'inégalité $c_{a\mu} \leqslant a\mathcal{J}_\mu(\tilde{u}_n) - c(a^{5/4}-a)\mu^3$, dont la limite $n \to +\infty$ nous donne la condition de sous-homogénéité stricte $c_{a\mu} \leqslant ac_\mu - c(a^{5/4}-a)\mu^3 < ac_\mu$, pour tout $a \in (1,4]$, et donc pour tout $a > 1$.

1. Introduction

Buffoni [1,2] recently studied the functional

$$\mathcal{J}_\mu(u) = \mathcal{K}(u) + \frac{\mu^2}{\mathcal{L}(u)}, \quad u \in U \setminus \{0\}, \tag{1}$$

where $U = \{u \in H^2(\mathbb{R}):\ \|u\|_2 < R\}$,

$$\mathcal{K}(u) = \int_\mathbb{R} \left\{\beta\sqrt{\left(u'\right)^2 + (1+Nu)^2} - \beta(1+Nu) + \frac{1}{2}u^2(1+Nu)\right\}\mathrm{d}x, \qquad \mathcal{L}(u) = \frac{1}{2}\Lambda\int_\mathbb{R} uNu\,\mathrm{d}x,$$

$$(Nu)(s) = \mathcal{F}^{-1}\left[f(k)\hat{u}(k)\right], \qquad \Lambda = \inf\left\{\frac{1+\beta k^2}{f(k)}:\ k \in \mathbb{R}\right\} > 0, \quad f(k) = |k|\coth|k|$$

and β, μ are positive constants with $\mu \ll 1$; the infimum in the definition of Λ is attained at $k = 0$ for $\beta > 1/3$ and at $k = \pm\omega \neq 0$ for $\beta < 1/3$. The functionals $\mathcal{K}$ and $\mathcal{L}$ are analytic at the origin in $H^2(\mathbb{R})$, so that $\mathcal{J}_\mu$ is of class C^∞ for sufficiently small values of R. Its critical points correspond to small-amplitude solitary water waves with Bond number $\beta > 0$ and fluid domain $\{(\xi, y):\ y \in (-1, \eta(\xi))\}$ (in a dimensionless coordinate system moving with the wave), where $\eta(\xi) = u(x)$, $\xi = x + \int_0^x (Nu)(s)\,\mathrm{d}s$. Buffoni's results are summarised in the following theorem (a precise definition of 'conditional energetic stability' is given in his papers).

Theorem 1.1.

(i) *Suppose that any minimising sequence* $\{u_n\}$ *for* $\mathcal{J}_\mu$ *over* $U \setminus \{0\}$ *with* $\sup_n \|u_n\|_2 < R$ *converges* (*up to subsequences and translations*) *weakly in* $H^2(\mathbb{R})$ *and strongly in* $H^s(\mathbb{R})$ *for* $s \in [0,2)$. *The set of minimisers of* $\mathcal{J}_\mu$ *over* $U \setminus \{0\}$ *defines a conditionally energetically stable family of solitary waves.*

(ii) *Let* $c_\mu = \inf_{u \in U \setminus \{0\}} \mathcal{J}_\mu(u)$. *The convergence criterion in part* (i) *is met whenever the strict subhomogeneity property* $c_{a\mu} < ac_\mu$ *holds for each* $a > 1$.

This strict subhomogeneity property was established by Buffoni [1] for $\beta > 1/3$ (strong surface tension), but only partial results are available for $\beta < 1/3$ (Buffoni [2,3]). In this Note we establish the strict subhomogeneity property for $\beta < 1/3$ and thus complete the existence and stability theory for the corresponding solitary waves. Existence theories for solitary waves in this parameter regime have previously been given by Iooss and Kirchgässner [5] and Buffoni and Groves [4].

Let us therefore choose $\beta < 1/3$ and write $\mathcal{K}(u) = \mathcal{K}_2(u) + \mathcal{M}(u)$, $\mathcal{M}(u) = \mathcal{K}_3(u) + \mathcal{K}_4(u) + \mathcal{K}_\mathrm{r}(u)$, where $\mathcal{K}_j(u) = \frac{1}{j!}\,\mathrm{d}\mathcal{K}[0](\{u\}^{(j)})$ is the part of $\mathcal{K}(u)$ which is homogeneous of degree j in u. Examining the explicit formulae for $\mathcal{K}_j$, $\mathcal{K}_\mathrm{r}$ and

$\mathcal{L}$, one finds that their gradients $\mathcal{K}_j'$, $\mathcal{K}_r'$ and $\mathcal{L}'$ in $L^2(\mathbb{R})$ exist for each $u \in U$ and define functions $H^2(\mathbb{R}) \to L^2(\mathbb{R})$ which are analytic at the origin. It follows that

$$\mathcal{J}_\mu'(u) := \mathcal{K}'(u) - \frac{\mu^2}{\mathcal{L}(u)^2}\mathcal{L}'(u) \tag{2}$$

defines a smooth function $U \setminus \{0\} \to L^2(\mathbb{R})$. Further relevant properties of this minimisation problem are listed in Lemma 1.2 below (see Buffoni [2] for the proof); note that the minimising sequence in part (iii) satisfies $\mathcal{J}_\mu'(\tilde{u}_n) \to 0$ in $L^2(\mathbb{R})$, which implies the usual Palais–Smale condition $\mathrm{d}\mathcal{J}_\mu[\tilde{u}_n] \to 0$ in $(H^2(\mathbb{R}))^\star$.

Lemma 1.2.

(i) *The inequality* $\mathcal{K}_2(u) + \mu^2/\mathcal{L}(u) \geqslant 2\mu$ *holds for each* $u \in U \setminus \{0\}$.
(ii) *There exists a function* $u^\star \in U \setminus \{0\}$ *such that* $\mathcal{J}_\mu(u^\star) < 2\mu - c\mu^3$, *so that* $c_\mu < 2\mu - c\mu^3$.
(iii) *There exists a minimising sequence* $\{\tilde{u}_n\}$ *for* $\mathcal{J}_\mu$ *over* $U \setminus \{0\}$ *with the properties that* $\|\tilde{u}_n\|_2^2 \leqslant c\mu$ *and* $\|\mathcal{J}_\mu'(\tilde{u}_n)\|_0 \to 0$ *as* $n \to \infty$.

Complete proofs of the results in this Note are given by Groves and Wahlén [6,7], in which solitary waves on deep water and on the surface of more general flows with constant vorticity are treated.

2. Estimates for near minimisers

In this section we show that any function $u \in H^2(\mathbb{R})$ with the properties that $\|u\|_2^2 \leqslant c\mu$, $\mathcal{J}_\mu(u) < 2\mu$ and $\|\mathcal{J}_\mu'(u)\|_0 \leqslant \mu^N$ for some $N \geqslant 3$ satisfies certain additional, sharper estimates. We begin by writing Eq. (2) as

$$g(k)\hat{u} = \mathcal{F}\big[\mathcal{J}_\mu'(u) - \mathcal{M}'(u) + \mathcal{S}(u)\mathcal{L}'(u)\big], \qquad \mathcal{S}(u) = \frac{\mu^2}{\mathcal{L}(u)^2} - 1,$$

where $g(k) = 1 + \beta k^2 - \Lambda f(k)$; note that $g(k) \geqslant 0$ with equality at precisely the values $k = \pm\omega$.

Proposition 2.1. *Choose* $\delta \in (0, \omega/3)$. *The mapping* $f \mapsto \mathcal{F}^{-1}[\frac{1-\chi(k)}{g(k)}\hat{f}]$ *defines a bounded linear operator* $L^2(\mathbb{R}) \to H^2(\mathbb{R})$, *where* $\chi = \chi_{[-\omega-\delta,-\omega+\delta]} + \chi_{[\omega-\delta,\omega+\delta]}$ *and* χ_S *denote the characteristic function of the subset* S *of* $\mathbb{R}$.

Define

$$u_2 = \mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}\big[\mathcal{J}_\mu'(u) - \mathcal{M}'(u) + \mathcal{S}(u)\mathcal{L}'(u)\big]\right] \tag{3}$$

and $u_1 := u - u_2$, so that $u_2 \in H^2(\mathbb{R})$, $\operatorname{supp}\hat{u}_1 \subseteq [-\omega-\delta, -\omega+\delta] \cup [\omega-\delta, \omega+\delta]$ and $\chi(k)\mathcal{F}[\mathcal{K}_3'(u_1)] = 0$, $\mathcal{K}_3(u_1) = 0$; it follows that

$$g(k)\hat{u}_1 = \chi(k)\mathcal{F}\big[\mathcal{J}_\mu'(u) - \mathcal{M}'(u) + \mathcal{K}_3'(u_1) + \mathcal{S}(u)\mathcal{L}'(u)\big] \tag{4}$$

and

$$\underbrace{u_2 + \mathcal{G}(u_1)}_{:= u_3} = \mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}\big[\mathcal{J}_\mu'(u) - \mathcal{M}'(u) + \mathcal{K}_3'(u_1) + \mathcal{S}(u)\mathcal{L}'(u)\big]\right], \tag{5}$$

where

$$\mathcal{G}(u_1) = \mathcal{F}^{-1}\left[\frac{1}{g(k)}\mathcal{F}\big[\mathcal{K}_3'(u_1)\big]\right] = \mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}\big[\mathcal{K}_3'(u_1)\big]\right]. \tag{6}$$

(Writing $u = u_1 + u_2$ is reminiscent of the corresponding decomposition into a leading order 'centre' part and a higher-order 'hyperbolic' part in the centre-manifold theory used by Iooss and Kirchgässner [5] and Buffoni and Groves [4].) We proceed by estimating $\|\mathcal{G}(u_1)\|_2^2$, $\|u_3\|_2^2$ and

$$|||u_1|||_\alpha^2 := \int_{\mathbb{R}} \big(1 + \mu^{-4\alpha}|k-\omega|^4\big)\big|\hat{u}_1^+(k)\big|^2\,\mathrm{d}k + \int_{\mathbb{R}} \big(1 + \mu^{-4\alpha}|k+\omega|^4\big)\big|\hat{u}_1^-(k)\big|^2\,\mathrm{d}k,$$

where $\hat{u}_1^+ = \hat{u}_1\chi_{[0,\infty)}$, $\hat{u}_1^- = \hat{u}_1\chi_{(-\infty,0]}$; note that $\|u_1\|_2$, $\|\mathcal{G}(u_1)\|_2$, $\|u_3\|_2$ are all *a priori* $O(\mu^{\frac{1}{2}})$.

Proposition 2.2. *The estimates* $\|v_1\|_{1,\infty} \leqslant c\mu^{\frac{\alpha}{2}}|||v_1|||_\alpha$, $\|v_1'' + \omega^2 v_1\|_0 \leqslant c\mu^\alpha|||v_1|||_\alpha$, $\|Nv_1 - f(\omega)v_1\|_0 \leqslant c\mu^\alpha|||v_1|||_\alpha$ *hold for any function* $v_1 \in L^2(\mathbb{R})$ *with* $\operatorname{supp}\hat{v}_1 \subseteq [-\omega-\delta, \omega+\delta] \cup [\omega-\delta, \omega+\delta]$.

Lemma 2.1.

(i) $\|\mathcal{G}(u_1)\|_2 \leqslant c\mu^{\frac{\alpha}{2}} |||u_1|||_\alpha \|u_1\|_2$;
(ii) $\|\mathcal{K}_3'(u) - \mathcal{K}_3'(u_1)\|_0 \leqslant c(\mu^{\frac{1}{2}+\alpha} |||u_1|||_\alpha^2 + \mu^{\frac{1}{2}} \|u_3\|_2)$;
(iii) $|\mathcal{K}_3(u)| \leqslant c(\mu^{1+\alpha} |||u_1|||_\alpha^2 + \mu\|u_3\|_2)$, $|\mathcal{K}_3(u)| \leqslant c(\mu^{\frac{3}{2}+\frac{\alpha}{2}} |||u_1|||_\alpha + \mu\|u_3\|_2)$.

Proof. These inequalities are obtained by writing $\mathcal{K}_3'(u) = m(\{u_1 - \mathcal{G}(u_1) + u_3\}^{(2)})$ and $\mathcal{K}_3(u) = n(\{u_1 - \mathcal{G}(u_1) + u_3\}^{(3)})$, where $m: H^2(\mathbb{R}) \times H^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $n: H^2(\mathbb{R}) \times H^2(\mathbb{R}) \times H^2(\mathbb{R}) \to \mathbb{R}$ are explicitly known multilinear operators, and estimating the five terms in $K_3'(u) - K_3'(u_1)$ and nine terms in $\mathcal{K}_3(u) = \mathcal{K}_3(u) - \mathcal{K}_3(u_1)$ using the rules given in Proposition 2.2. □

Lemma 2.2. *The estimates*

$$\|\mathcal{K}_4'(v)\|_0 \leqslant c(\|v'\|_\infty + \|Nv\|_\infty)^2 \|v\|_2,$$
$$|\mathcal{K}_{\mathrm{r}}(v)| \leqslant c(\|v'\|_\infty + \|Nv\|_\infty)^3 \|v\|_2^2, \qquad \|\mathcal{K}_{\mathrm{r}}'(v)\|_0 \leqslant c(\|v'\|_\infty + \|Nv\|_\infty)^3 \|v\|_2$$

hold for each $v \in U$.

Corollary 2.3.

(i) $\|\mathcal{M}'(u) - \mathcal{K}_3'(u_1)\|_0 \leqslant c(\mu^{\frac{1}{2}+\alpha} |||u_1|||_\alpha^2 + \mu^{\frac{1}{2}} \|u_3\|_2)$;
(ii) $|\mathcal{M}(u)| \leqslant c(\mu^{1+\alpha} |||u_1|||_\alpha^2 + \mu\|u_3\|_2)$, $|\mathcal{M}(u)| \leqslant c(\mu^{\frac{3}{2}+\frac{\alpha}{2}} |||u_1|||_\alpha + \mu\|u_3\|_2)$;
(iii) $|\langle\mathcal{M}'(u), u\rangle| \leqslant c(\mu^{1+\alpha} |||u_1|||_\alpha^2 + \mu\|u_3\|_2)$, $|\langle\mathcal{M}'(u), u\rangle| \leqslant c(\mu^{\frac{3}{2}+\frac{\alpha}{2}} |||u_1|||_\alpha + \mu\|u_3\|_2)$.

Proof. Observe that

$$\mathcal{M}'(u) - \mathcal{K}_3'(u_1) = (\mathcal{K}_3'(u) - \mathcal{K}_3'(u_1)) + \mathcal{K}_4'(u) + \mathcal{K}_{\mathrm{r}}'(u),$$
$$\mathcal{M}(u) = \mathcal{K}_3(u) + \frac{1}{4}\langle\mathcal{K}_4'(u), u\rangle + \mathcal{K}_{\mathrm{r}}(u),$$
$$|\langle\mathcal{M}'(u), u\rangle| \leqslant 3|\mathcal{K}_3(u)| + |\langle\mathcal{K}_4'(u), u\rangle| + \|\mathcal{K}_{\mathrm{r}}'(u)\|_0 \|u\|_2$$

and apply the estimates given in Lemmata 2.1 and 2.2. □

Lemma 2.4. $|\mathcal{S}(u)| \leqslant c(\mu^\alpha |||u_1|||_\alpha^2 + \|u_3\|_2 + \mu^{N-\frac{1}{2}})$.

Proof. Writing Eq. (2) as

$$\frac{\mu}{\mathcal{L}(u)} = -\frac{\langle\mathcal{J}_\mu'(u), u\rangle}{2\mu} + \frac{\mathcal{K}_2(u)}{\mu} + \frac{\langle\mathcal{M}'(u), u\rangle}{2\mu}$$

and using the inequalities $\mathcal{K}_2(u) + \mu^2/\mathcal{L}(u) + \mathcal{M}(u) \leqslant 2\mu \leqslant \mathcal{K}_2(u) + \mu^2/\mathcal{L}(u)$, we find that $2\mathcal{R}_1(u) + \mathcal{R}_1(u)^2 \leqslant \mathcal{S}(u) \leqslant 2\mathcal{R}_2(u) + \mathcal{R}_2(u)^2$, where

$$\mathcal{R}_1(u) = -\frac{\langle\mathcal{J}_\mu'(u), u\rangle}{4\mu} + \frac{\langle\mathcal{M}'(u), u\rangle}{4\mu}, \qquad \mathcal{R}_2(u) = -\frac{\langle\mathcal{J}_\mu'(u), u\rangle}{4\mu} + \frac{\langle\mathcal{M}'(u), u\rangle}{4\mu} - \frac{\mathcal{M}(u)}{2\mu}.$$

The result is obtained by estimating $\mathcal{R}_1(u)$, $\mathcal{R}_2(u)$ by means of Corollary 2.3. □

Theorem 2.5. *The inequalities $|||u_1|||_\alpha^2 \leqslant c\mu$, $\|u_3\|_2^2 \leqslant c\mu^{3+2\alpha}$ and $\|\mathcal{G}(u_1)\|_2^2 \leqslant c\mu^{2+\alpha}$ hold for each $\alpha < 1$.*

Proof. The inequality

$$\|u_3\|_2^2 \leqslant c(\mu^{1+2\alpha} |||u_1|||_\alpha^4 + \mu^{2N}) \tag{7}$$

is obtained by estimating the right-hand side of Eq. (5) using Corollary 2.3 and Lemma 2.4. Using this inequality and the facts that $|g(k)| \geqslant c|k-\omega|^4$ for $k \in (\omega-\delta, \omega+\delta)$ and $|g(k)| \geqslant c|k+\omega|^4$ for $k \in (-\omega-\delta, -\omega+\delta)$, one finds from Eq. (4) that

$$\begin{aligned} |||u_1|||_\alpha^2 &\leqslant c(\mu^{-4\alpha}(\|\mathcal{J}_\mu'(u)\|_0^2 + |\mathcal{S}(u)|^2 \|\mathcal{L}'(u)\|_0^2 + \|\mathcal{M}'(u) - \mathcal{K}_3'(u)\|_0^2) + \|u_1\|_0^2) \\ &\leqslant c(\mu^{-4\alpha}(\mu^{1+2\alpha} |||u_1|||_\alpha^4 + \mu\|u_3\|_2^2 + \mu^{2N}) + \mu) \leqslant c(\mu^{1-2\alpha} |||u_1|||_\alpha^4 + \mu), \end{aligned}$$

where we have estimated $\|\mathcal{L}'(u)\|_0^2 \leqslant c\|u\|_2^2 \leqslant c\mu$ and taken $\alpha \leqslant \frac{N}{2} - \frac{1}{4}$; it follows that

$$\frac{|||u_1|||_\alpha^2}{\mu} \leqslant c\left(1+\mu^{2-2\alpha}\left(\frac{|||u_1|||_\alpha^2}{\mu}\right)^2\right).$$

Define $M = \{\alpha \in (-\infty, 1)\colon\ |||u_1|||_\alpha^2 \leqslant c\mu\}$. The inequality $|||u_1|||_{\alpha_1}^2 \leqslant |||u_1|||_{\alpha_2}^2$ for $\alpha_1 \leqslant \alpha_2$ shows that $(-\infty, \alpha] \subset M$ whenever $\alpha \in M$; furthermore $(-\infty, 0] \subseteq M$ because $|||u_1|||_0^2 \leqslant \|u_1\|_0^2 \leqslant c\mu$. Suppose that $\alpha^\star := \sup M$ is strictly less than unity, choose $\varepsilon > 0$ so that $\alpha^\star + 9\varepsilon < 1$ and observe that

$$\frac{|||u_1|||_{\alpha^\star+\varepsilon}^2}{\mu} \leqslant c\left(1+\mu^{2-2\alpha^\star-18\varepsilon}\underbrace{\left(\frac{|||u_1|||_{\alpha^\star-\varepsilon}^2}{\mu}\right)^2}_{\leqslant c}\right) \leqslant c,$$

which leads to the contradiction that $\alpha^\star + \varepsilon \in M$. It follows that $\alpha^\star = 1$ and $|||u_1|||_\alpha^2 = O(\mu)$ for each $\alpha < 1$. The estimates for $\mathcal{G}(u_1)$ and u_3 follow directly from this result, Lemma 2.1 and inequality (7). □

3. Estimates for the variational functional

The next step is to identify the dominant term in $\mathcal{M}(u)$, where $u \in H^2(\mathbb{R})$ is a function with the properties that $\|u\|_2^2 \leqslant c\mu$, $\mathcal{J}_\mu(u) < 2\mu - c\mu^3$ and $\|\mathcal{J}_\mu'(u)\|_0 \leqslant \mu^N$; to this end we use Theorem 2.5 and choose $\alpha > 1/2$. Our first result is obtained by combining the estimates in this theorem with those in Lemmata 2.1 and 2.2.

Lemma 3.1.

(i) $\mathcal{K}_{\mathrm{r}}(u) = o(\mu^3)$;
(ii) $\mathcal{K}_4(u) = \mathcal{K}_4(u_1) + o(\mu^3)$;
(iii) $\mathcal{K}_3(u) = -\int_{\mathbb{R}} \mathcal{K}_3'(u_1)\mathcal{G}(u_1)\,\mathrm{d}x + o(\mu^3)$.

Proposition 3.1.

(i) $N(u_1^\pm) = f(\omega)u_1^\pm + \underline{O}(\mu^{\frac{1}{2}+\alpha})$, $(u_1^\pm)' = \pm \mathrm{i}\omega u_1^\pm + \underline{O}(\mu^{\frac{1}{2}+\alpha})$;
(ii) $((u_1^\pm)^2)' = \pm 2\mathrm{i}\omega(u_1^\pm)^2 + \underline{O}(\mu^{1+\frac{3\alpha}{2}})$, $N((u_1^\pm)^2) = f(2\omega)(u_1^\pm)^2 + \underline{O}(\mu^{1+\frac{3\alpha}{2}})$;
(iii) $N(u_1^+u_1^-) = u_1^+u_1^- + \underline{O}(\mu^{1+\frac{3\alpha}{2}})$;
(iv) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[(u_1^\pm)^2]] = g(2\omega)^{-1}(u_1^\pm)^2 + \underline{O}(\mu^{1+\frac{3\alpha}{2}})$, $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[u_1^+u_1^-]] = g(0)^{-1}u_1^+u_1^- + \underline{O}(\mu^{1+\frac{3\alpha}{2}})$.

Here the symbol $\underline{O}(\mu^\gamma)$ denotes a quantity whose $H^1(\mathbb{R})$-norm is $O(\mu^\gamma)$.

The next lemma is proved by substituting $u_1 = u_1^- + u_1^+$ into the estimates in Lemma 3.1 and approximating the operators $N(\cdot)$ and $(\cdot)'$ using the rules given in Proposition 3.1; note in particular that the leading-order part of $\mathcal{K}_3(u)$ is actually a quartic function of u_1.

Lemma 3.2.

$$\mathcal{K}_3(u) = A_1(\omega)\int_{\mathbb{R}} u_1^4\,\mathrm{d}x + o(\mu^3), \qquad \mathcal{K}_4(u) = A_2(\omega)\int_{\mathbb{R}} u_1^4\,\mathrm{d}x + o(\mu^3),$$

where

$$A_1(\omega) = -\frac{1}{12}g(2\omega)^{-1}\left(2f(\omega) + f(2\omega) - 4\beta\omega^2 f(\omega) + \beta\omega^2 f(2\omega)\right)^2 - \frac{1}{6}g(0)^{-1}\left(2f(\omega) + 1 - \beta\omega^2\right)^2,$$
$$A_2(\omega) = \frac{1}{24}\left(-3\beta\omega^4 + 4\beta\omega^2 f(\omega)^2\right) > 0.$$

Corollary 3.3. $\int_{\mathbb{R}} u_1^4\,\mathrm{d}x > c\mu^3$.

Proof. The inequalities $\mathcal{K}_2(u) + \mu^2/\mathcal{L}(u) + \mathcal{M}(u) < 2\mu - c\mu^3$ and $\mathcal{K}_2(u) + \mu^2/\mathcal{L}(u) \geqslant 2\mu$ show that $\mathcal{M}(u) < -c\mu^3$, and the result (together with the inequality $A_1(\omega) + A_2(\omega) < 0$) follows from this observation and the estimate $\mathcal{M}(u) = (A_1(\omega) + A_2(\omega))\int_{\mathbb{R}} u_1^4\,\mathrm{d}x + o(\mu^3)$ (see Lemmata 3.1 and 3.2). □

4. Strict subhomogeneity

The strict subhomogeneity criterion may now be deduced by applying the estimates recorded above to the sequence $\{\tilde{u}_n\}$ identified in Lemma 1.2(iii).

Proposition 4.1. $A_1(\omega) + 6A_2(\omega) < 0$.

Proof. Using the relations $\beta = f'(\omega)/(2\omega f(\omega) - \omega^2 f'(\omega))$, $\Lambda = 2\omega/(2\omega f(\omega) - \omega^2 f'(\omega))$, we obtain an explicit formula for $A_1(\omega) + 6A_2(\omega)$. □

Lemma 4.1. *The function $a \mapsto a^{-5/2}\mathcal{M}(a\tilde{u}_n)$, $a \in [1,2]$, is decreasing and strictly negative.*

Proof. Observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\big(a^{-5/2}\mathcal{M}(a\tilde{u}_n)\big) &= -\frac{5}{2}a^{-7/2}\mathcal{M}(a\tilde{u}_n) + a^{-5/2}\big\langle \mathcal{M}'(a\tilde{u}_n), \tilde{u}_n\big\rangle_0 = a^{-7/2}\bigg(-\frac{5}{2}\mathcal{M}(a\tilde{u}_n) + \big\langle \mathcal{M}'(a\tilde{u}_n), a\tilde{u}_n\big\rangle_0\bigg) \\
&= a^{-7/2}\bigg(\frac{1}{2}\mathcal{K}_3(a\tilde{u}_n) + \frac{3}{2}\mathcal{K}_4(a\tilde{u}_n) \underbrace{-\frac{5}{2}\mathcal{K}_{\mathrm{r}}(a\tilde{u}_n) + \big\langle \mathcal{K}_{\mathrm{r}}(a\tilde{u}_n), a\tilde{u}_n\big\rangle}_{\substack{= O((\|a\tilde{u}_n'\|_\infty + \|N(a\tilde{u}_n)\|_\infty)^3\|a\tilde{u}_n\|_2^2) \\ = a^5\, O((\|\tilde{u}_n'\|_\infty + \|N\tilde{u}_n\|_\infty)^3\|\tilde{u}_n\|_2^2) \\ = a^5 o(\mu^3)}}\bigg) \\
&= \frac{1}{2}a^{-1/2}\big(\mathcal{K}_3(\tilde{u}_n) + 3a\mathcal{K}_4(\tilde{u}_n)\big) + o\big(\mu^3\big) = \frac{1}{2}a^{-1/2}\big(A_1(\omega) + 3aA_2(\omega)\big)\int_{\mathbb{R}} \tilde{u}_{n,1}^4\,\mathrm{d}x + o\big(\mu^3\big) \\
&\leqslant \frac{1}{2}a^{-1/2}\big(A_1(\omega) + 6A_2(\omega)\big)\int_{\mathbb{R}} \tilde{u}_{n,1}^4\,\mathrm{d}x + o\big(\mu^3\big) < -c\mu^3 + o\big(\mu^3\big) < 0
\end{aligned}$$

for $a \in [1,2]$. □

Theorem 4.2. *The strict subhomogeneity criterion $c_{a\mu} < ac_\mu$ holds for each $a > 1$.*

Proof. It suffices to establish this result for $a \in (1,4]$ (see Buffoni [1, p. 56]). Replacing a by $a^{1/2}$, we find from Lemma 4.1 that $\mathcal{M}(a^{1/2}\tilde{u}_n) \leqslant a^{5/4}\mathcal{M}(\tilde{u}_n)$ for $a \in (1,4]$ and therefore that

$$\begin{aligned}
c_{a\mu} &\leqslant \mathcal{K}_2\big(a^{1/2}\tilde{u}_n\big) + \frac{a^2\mu^2}{\mathcal{L}(a^{1/2}\tilde{u}_n)} + \mathcal{M}\big(a^{1/2}\tilde{u}_n\big) \leqslant a\bigg(\mathcal{K}_2(\tilde{u}_n) + \frac{\mu^2}{\mathcal{L}(\tilde{u}_n)}\bigg) + a^{5/4}\mathcal{M}(\tilde{u}_n) \\
&= a\bigg(\mathcal{K}_2(\tilde{u}_n) + \frac{\mu^2}{\mathcal{L}(\tilde{u}_n)} + \mathcal{M}(\tilde{u}_n)\bigg) + \big(a^{5/4} - a\big)\mathcal{M}(\tilde{u}_n) \leqslant a\mathcal{J}_\mu(\tilde{u}_n) - c\big(a^{5/4} - a\big)\mu^3
\end{aligned}$$

for $a \in (1,4]$. In the limit $n \to \infty$ the above inequality yields $c_{a\mu} \leqslant ac_\mu - c(a^{5/4} - a)\mu^3 < ac_\mu$. □

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