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A note on statistical solutions of the three-dimensional Navier–Stokes equations: The stationary case

Note sur les solutions statistiques des équations de Navier–Stokes incompressibles en dimension trois d'espace: le cas stationnaire

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ABSTRACT

Stationary statistical solutions of the three-dimensional Navier–Stokes equations for incompressible fluids are considered. They are a mathematical formalization of the notion of ensemble average for turbulent flows in statistical equilibrium in time. They are also a generalization of the notion of invariant measure to the case of the three-dimensional Navier–Stokes equations, for which a global uniqueness result is not known to exist and a semigroup may not be well-defined in the classical sense. The two classical definitions of stationary statistical solutions are considered and compared, one of them being a particular case of the other and possessing a number of useful properties. Furthermore, the so-called time-average stationary statistical solutions, obtained as generalized limits of time averages of weak solutions as the averaging time goes to infinity are shown to belong to this more restrictive class. A recurrent type result is also obtained for statistical solutions satisfying an accretion condition. Finally, the weak global attractor of the three-dimensional Navier–Stokes equations is considered, and in particular it is shown that there exists a topologically large subset of the weak global attractor which is of full measure with respect to that particular class of stationary statistical solutions and which has a certain regularity property.

RÉSUMÉ

Dans cette Note nous considérons les solutions statistiques stationnaires des équations de Navier–Stokes pour les fluides incompressibles. Les solutions statistiques stationnaires sont la formulation mathématique du concept de moyenne statistique pour des écoulements turbulents qui sont en équilibre statistique dans le temps. Ce sont aussi des généralisations de la notion de mesure invariante pour les équations de Navier–Stokes tridimensionnelles. En effet l'existence globale en temps d'une solution régulière de ces équations n'étant pas connue, on ne peut leur associer un semi-groupe d'opérateurs continus avec la définition qui en résulterait d'une solution mesure invariante. Deux définitions classiques des solutions statistiques stationnaires ont eté proposées; elles sont ici rappelées et comparées ; les premières sont un cas particulier des secondes, et elles possèdent plusieurs propriétés utiles. De plus nous considérons les solutions statistiques stationnaires qui sont obtenues comme moyennes en temps de solutions faibles évolutives sur des intervalles

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de temps de plus en plus grands et nous montrons que ces moyennes temporelles appartiennent à la plus petite classe de solutions statistiques stationnaires. En outre une propriété de type récurrence est obtenue pour les solutions statistiques stationnaires qui satisfont une propriété d'accrétivité définie dans le texte. Finalement nous nous intéressons à l'attracteur global faible associé aux équations de Navier–Stokes tridimensionnelles, et nous montrons en particulier qu'il existe un sous-ensemble « topologiquement grand » de cet attracteur dont la mesure est totale par rapport à cette classe particulière de solutions statistiques stationnaires et qui présente un certain caractère de régularité.

Version française abrégée

Dans cette Note nous nous intéressons aux solutions statistiques stationnaires des équations de Navier–Stokes incompressibles en dimension d'espace trois ; cette Note fait suite à une note précédente, [3], dans laquelle nous considérions les solutions statistiques évolutives. Le lecteur intéressé pourra consulter [3] pour des précisions complémentaires, mais nous nous proposons dans la présente Note de faire une présentation autonome de nos résultats au prix de quelques répétitions avec [3].

Les solutions statistiques [2,8,20,21,7,3] ont été introduites pour formaliser la notion de moyenne statistique qui est au centre de la théorie conventionnelle de la turbulence [12,16,18,11,14]. Une solution statistique est une famille de mesures qui dépendent de la variable temporelle, et qui décrivent la distribution de probabilité du champ des vitesses à chaque instant. Les solutions statistiques évolutives correspondent à des écoulements turbulents pour lesquels l'information statistique varie avec le temps, ce qui est par exemple le cas en turbulence décroissante, ou pour l'écoulement en aval d'une grille. Il existe par contre de nombreuses situations où l'écoulement est en équilibre statistique temporel, par exemple les écoulements dans un conduit avec un grand gradient de pression ; pour de telles situations, la distribution de probabilité du champ des vitesses ne varie pas au cours du temps (bien que l'écoulement qu'elle décrit dépende lui-même du temps), et la solution statistique associée est alors stationnaire.

Dans la première Note [3], nous avons considéré la formulation mathématique des deux principales notions de solutions statistiques, introduites respectivement par Foias et Prodi [2,8] et par Vishik et Fursikov [20]. Nous avons proposé une nouvelle formulation de solutions statistiques semblable mais non identique à la formulation de Vishik et Fursikov, que nous avons appelées solutions statistiques de Vishik–Fursikov. Il s'avère, avec cette définition légèrement modifiée, que toute solution statistique au sens de Vishik–Fursikov est une solution statistique au sens de Foias et Prodi, mais les solutions de Vishik–Fursikov se prêtent mieux à l'analyse que celles de Foias et Prodi et elles possèdent de nombreuses propriétés utiles. Dans cette Note nous étudions la version stationnaire de ces deux concepts.

Un type particulier de solutions statistiques stationnaires est obtenu comme la limite (au sens des limites généralisées) des moyennes en temps de solutions faibles des équations de Navier–Stokes tridimensionnelles, quand l'intervalle temporel de moyenne tend vers l'infini. Ces solutions statistiques moyennes temporelles sont importantes en raison de leur lien avec la théorie conventionnelle de la turbulence et la notion d'ergodicité. Nous démontrons ainsi que toute solution statistique stationnaire obtenue par moyenne temporelle appartient en fait à la classe la plus restreinte des solutions statistiques stationnaires, celles de Vishik–Fursikov, et elles héritent ainsi de propriétés utiles de ces dernières (voir les Théorèmes 4.1 et 4.2 de la version anglaise).

Nous montrons aussi que toute solution statistique stationnaire de Vishik–Fursikov est portée par l'ensemble $\mathcal{W}$ qui est formé de l'ensemble des trajectoires définies pour tout t, et bornées en temps sur tout $\mathbb{R}$ – (voir (3)).

En raison de l'absence, en dimension trois, d'un résultat d'existence de solutions régulières définies pour tout temps, il n'est pas possible de définir un semi-groupe dans l'espace des phases qui est l'espace H défini ci-dessous, et on ne peut utiliser la définition classique de mesure invariante pour le flot. De ce point de vue, la notion de mesure statistique stationnaire est aussi une sorte de généralisation dans ce contexte de la notion de mesure invariante pour le flot. On peut alors se demander dans quelle mesure des résultats de théorie ergodique peuvent être étendus ou adaptés aux solutions statistiques stationnaires. Dans cette direction nous présentons dans le Théorème 4.3 un résultat de type récurrence, toutefois plus faible que le classique Théorème de Récurrence de Poincaré. Par contre, et comme nous l'avons déjà indiqué, les solutions statistiques stationnaires de Vishik–Fursikov possèdent un certain nombre de propriétés supplémentaires et, en particulier, un résultat de récurrence, beaucoup plus proche du Théorème de Récurrence de Poincaré, est obtenu pour celles-ci.

Finalement nous étudions la structure de l'attracteur faible global $\mathcal{A}_{\mathrm{w}}$ introduit en [10]. Il est bien connu que toute trajectoire de $\mathcal{A}_{\mathrm{w}}$ est régulière sur un ouvert dense de $\mathbb{R}$ (de même que toute trajectoire faible est continue sur un sous ensemble ouvert de son ensemble temporel de définition). Dans [10] un ensemble $\mathcal{A}_{\text{rég}} \subset \mathcal{A}_{\mathrm{w}}$ a été introduit qui est fait de points de V qui appartiennent à une trajectoire (complète bornée) régulière dans le voisinage (temporel) de ce point et tels que, en outre, il n'y ait pas d'autre trajectoire complète bornée faible qui y aboutisse (ou qui en parte). Nous considérons aussi un autre sous-ensemble « topologiquement grand » de $\mathcal{A}_{\mathrm{w}}$, appelé $\mathcal{A}'_{\text{rég}}$ qui est constitué des conditions initiales telles qu'il existe au moins une solution faible qui admette ce point comme condition initiale et qui soit régulière dans un voisinage de l'origine. Nous montrons que toute solution statistique stationnaire de Vishik–Fursikov est portée par cet ensemble et de même par un sous-ensemble $\mathcal{W}'_{\text{rég}}$ de $\mathcal{W}$ défini ci-dessous (voir les Théorèmes 5.1 et 5.2). Ces résultats donnent une réponse partiellement positive à une conjecture de G. Prodi selon laquelle les solutions statistiques stationnaires devraient être portées par un ensemble de solutions initiales produisant des solutions globales régulières des équations de Navier–Stokes. Cette conjecture de G. Prodi constitue une sorte de résultat de régularité « presque partout » pour les solutions des équations de Navier–Stokes tridimensionnelles.

Nous renvoyons le lecteur aux articles à paraitre [4–6] pour les démonstrations de tous ces résultats et pour des détails supplémentaires.

1. Introduction

We consider stationary statistical solutions of the three-dimensional incompressible Navier–Stokes equations. This Note is a continuation of [3], in which time-dependent statistical solutions were considered. We refer the reader to [3] for further background, although we try to make this Note self-contained at the cost of some notational repetitions.

Statistical solutions [2,8,20,21,7,3] have been introduced to formalize the notion of ensemble average, which is so pervasive in the conventional theory of turbulence (see e.g. [16,18,12,11,14]). A statistical solution is a family of measures parametrized by the time variable and associated with the probability distribution of the velocity field of the flow at each time. Time-dependent statistical solutions apply to turbulent flows in which the statistical information varies with time, such as in decaying turbulence, following a flow past a grid, for example. There are many situations, however, in which the flow is in statistical equilibrium in time, such as flows in a pipe with a large pressure gradient; in those cases, the probability distribution of the velocity field does not change with time, and the corresponding ensemble average is formally associated to stationary statistical solutions.

In the first note [3], we reconsidered the mathematical formulation of the two main definitions of statistical solutions, introduced by Foias and Prodi [2,8] and by Vishik and Fursikov [20], respectively. We proposed a new formulation of statistical solutions inspired by the approach of Vishik–Fursikov and termed them the Vishik–Fursikov statistical solutions. Any Vishik–Fursikov statistical solution is a statistical solution in the sense of Foias and Prodi, but they are more amenable to analysis than the latter and possess a number of useful properties. Here we consider the stationary version of both types of statistical solutions.

A particular type of stationary statistical solution is obtained as the generalized limit of time averages of individual weak solutions of the three-dimensional Navier–Stokes equations as the averaging time goes to infinity. It is a particularly important type of statistical solution due to its connection to the conventional theory of turbulence and ergodic theory through the concept of time average. We obtain here that any so-called time-average stationary statistical solution is a Vishik–Fursikov stationary statistical solution, and hence it inherits a number of useful properties (see Theorems 4.1 and 4.2).

We also obtain that any Vishik–Fursikov stationary statistical solution is carried by the set $\mathcal{W}$ of initial data through which there exists a global weak solution uniformly bounded in time (for all times in $\mathbb{R}$).

Due to the lack of a known global uniqueness result for the three-dimensional Navier–Stokes equations, it is not possible to define a semigroup in phase space nor a concept of invariant measures in the classical sense. In that regard, stationary statistical solutions are somewhat generalizations of invariant measures to this context. It is therefore natural to ask whether the results in ergodic theory can be extended, or adapted, to statistical solutions. In that direction, we present a recurrence-type result which is slightly weaker than the classical Poincaré's Recurrence Theorem (see Theorem 4.3). On the other hand, as mentioned above, the Vishik–Fursikov stationary statistical solutions have a number of special properties, and, in particular, a result much closer to the classical Poincaré Recurrence Theorem is obtained.

Finally, we study the structure of the weak global attractor $\mathcal{A}_{\mathrm{w}}$ introduced in [10]. It is well known that any trajectory in $\mathcal{A}_{\mathrm{w}}$ is regular on a dense open subset of $\mathbb{R}$ (and more generally any weak trajectory is regular on a dense open subset of its temporal set of definition). In [10] a set $\mathcal{A}_{\mathrm{reg}} \subset \mathcal{A}_{\mathrm{w}}$ was introduced which is made of points in V which belong to a complete bounded trajectory which is regular on a neighborhood of that point on the trajectory, and such that, furthermore, there is no other complete bounded weak trajectory ending at or starting from this point. We also consider another topologically large subset $\mathcal{A}'_{\mathrm{reg}}$ of $\mathcal{A}_{\mathrm{w}}$ containing $\mathcal{A}_{\mathrm{reg}}$ and made of initial conditions such that there exists at least one weak solution with that initial condition and which is regular on a neighborhood of the initial time. We show then that any Vishik–Fursikov stationary statistical solution is carried by $\mathcal{A}'_{\mathrm{reg}}$, and similarly for a subset $\mathcal{W}'_{\mathrm{reg}}$ of $\mathcal{W}$ defined below (see Theorems 5.1 and 5.2). These results are a partial positive answer to a conjecture of Prodi [15] which essentially says that stationary statistical solutions should be carried by sets of initial conditions associated with global strong solutions of the Navier–Stokes equations, a sort of regularity result saying that almost all solutions of the three-dimensional Navier–Stokes equations are regular.

More details on the results presented here will be given in [4–6].

2. Preliminaries

2.1. The Navier–Stokes equations and the mathematical setting

We consider the three-dimensional Navier–Stokes equation with either no-slip or periodic boundary conditions. More precisely, we consider either flows on an open, bounded, regular set Ω in $\mathbb{R}^3$ with no-slip boundary conditions (Dirichlet conditions), or periodic flows in $\mathbb{R}^3$ with periods L_1, L_2, L_3 in each direction $0x_1, 0x_2, 0x_3$, respectively, and in this case we set Ω to be the period $\Omega = \prod_{i=1}^{3}(0, L_i)$.

In either case, the three-dimensional Navier–Stokes equation admits the following concise functional formulation [19,13]:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \tag{1}$$

in an appropriate functional space H. This space is defined as the completion in $L^2(\Omega)^3$ of

$$\mathcal{V}_{\mathrm{Dir}} = \left\{\mathbf{u} \in \mathcal{C}_c^\infty\left(\mathbb{R}^3\right)^3;\ \nabla \cdot \mathbf{w} = 0\right\}$$

in the Dirichlet case, and

$$\mathcal{V}_{\mathrm{per}} = \left\{\mathbf{u} = \mathbf{w}|_\Omega;\ \begin{array}{l}\mathbf{w} \in \mathcal{C}^\infty(\mathbb{R}^3)^3,\ \nabla \cdot \mathbf{w} = 0,\ \int_\Omega \mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,\ \mathbf{w}(\mathbf{x}) \text{ is periodic} \\ \text{with period } L_i \text{ in each direction } 0x_i\end{array}\right\}$$

in the periodic case.

We also consider the space V which is the completion of either $\mathcal{V}_{\mathrm{Dir}}$ or $\mathcal{V}_{\mathrm{per}}$ in $H^1(\Omega)^3$, depending on the case. The spaces H and V are endowed with the usual inner products

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \big((\mathbf{u}, \mathbf{v})\big) = \int_\Omega \sum_{i=1,2,3} \frac{\partial \mathbf{u}}{\partial x_i} \cdot \frac{\partial \mathbf{v}}{\partial x_i}\,\mathrm{d}\mathbf{x},$$

and the associated norms $|\mathbf{u}| = (\mathbf{u}, \mathbf{u})^{1/2}$, $\|\mathbf{u}\| = ((\mathbf{u}, \mathbf{u}))^{1/2}$.

The forcing term $\mathbf{f}$ is assumed to be time independent and to belong to H.

We denote by P_{LH} the (Leray–Helmhotz) orthogonal projector in $L^2(\Omega)^3$ onto the subspace H. In the Dirichlet case, the operator A in (1) is the Stokes operator given by $A\mathbf{u} = -P_{\mathrm{LH}}\Delta\mathbf{u}$, for $\mathbf{u} \in D(A)$, with domain $D(A) = V \cap H^2(\Omega)^3$. In the periodic case, A is given by $A\mathbf{u} = -P_{\mathrm{LH}}\Delta\mathbf{u} = -\Delta\mathbf{u}$, for $\mathbf{u} \in D(A)$, with the domain $D(A)$ as the closure in $H^2(\Omega)^3$ of $\mathcal{V}_{\mathrm{per}}$.

Finally, the term $B(\mathbf{u}, \mathbf{v}) = P_{\mathrm{LH}}((\mathbf{u} \cdot \nabla)\mathbf{v})$ is a bilinear term associated with the inertial term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ of the Navier–Stokes equations.

2.2. Weak and strong solutions

We call a function $\mathbf{u} = \mathbf{u}(t)$ on a time interval $I \subset \mathbb{R}$ with values in H a *(Leray–Hopf) weak solution* of (1) on I if (i) $\mathbf{u} \in L^\infty_{\mathrm{loc}}(I; H) \cap L^2_{\mathrm{loc}}(I; V)$; (ii) $t \mapsto \mathbf{u}(t)$ is weakly continuous in H; (iii) $\mathbf{u}$ satisfies the functional equation (1) in the distribution sense on I, with values in the dual V' of V; (iv) for almost all t' in I, $\mathbf{u}$ satisfies the energy inequality

$$\frac{1}{2}\left|\mathbf{u}(t)\right|^2 + \nu \int_{t'}^{t} \left\|\mathbf{u}(s)\right\|^2 \mathrm{d}s \leqslant \frac{1}{2}\left|\mathbf{u}(t')\right|^2 + \int_{t'}^{t} \big(\mathbf{f}, \mathbf{u}(s)\big)\,\mathrm{d}s, \tag{2}$$

for all t in I with $t > t'$; and (v) if I is closed and bounded on the left, with left end point denoted by t_0, then $\mathbf{u}$ is strongly continuous in H at t_0 from the right, i.e. $\mathbf{u}(t) \to \mathbf{u}(t_0)$ in H as $t \to t_0^+$. The times t' allowed in (2) are the points of strong continuity from the right for $\mathbf{u}$, and their set is of total measure in I. We recall that for any interval $I = [t_0, \infty)$ and any $\mathbf{u}_0 \in H$, there exists a Leray–Hopf weak solution on I such that $\mathbf{u}(0) = \mathbf{u}_0$.

A weak solution is called a *strong solution* on an interval I when it is continuous on I as a function with values in V.

2.3. Trajectory spaces

Let I be an arbitrary interval in $\mathbb{R}$, H_{w} denote the space H endowed with its weak topology, and $\mathcal{C}(I, H_{\mathrm{w}})$ be the space of continuous functions from I into H_{w} endowed with the topology of uniform weak convergence on compact intervals in I. Denoting by I° the interior of I, we also define the spaces of weak solutions $\mathcal{U}_I = \{\mathbf{u} \in \mathcal{C}(I, H_{\mathrm{w}});\ \mathbf{u}$ is a weak solution on $I\}$, and $\mathcal{U}_I^\sharp = \{\mathbf{u} \in \mathcal{C}(I, H_{\mathrm{w}});\ \mathbf{u}$ is a weak solution on $I^\circ\}$. In the case $I = \mathbb{R}$, we also consider the space

$$\mathcal{W} = \left\{\mathbf{u} \in \mathcal{C}(\mathbb{R}, H_{\mathrm{w}});\ \mathbf{u} \text{ is a weak solution on } \mathbb{R} \text{ uniformly bounded in } H\right\}. \tag{3}$$

For each $t_0 \in I$, we define the projection operator $\Pi_{t_0} : \mathcal{C}(I, H_{\mathrm{w}}) \to H_{\mathrm{w}}$ by setting $\Pi_{t_0}\mathbf{u} = \mathbf{u}(t_0)$. Similarly, for each subinterval $J \subset I$, we define the restriction operator $\Pi_J : \mathcal{C}(I, H_{\mathrm{w}}) \to \mathcal{C}(J, H_{\mathrm{w}})$ by $(\Pi_J\mathbf{u})(t) = \mathbf{u}(t)$, for all $t \in J$. Finally, for any interval I unbounded on the right, we define the time-translation operators $\sigma_\tau : \mathcal{C}(I, H_{\mathrm{w}}) \to \mathcal{C}(I, H_{\mathrm{w}})$ by $(\sigma_\tau\mathbf{u})(t) = \mathbf{u}(t + \tau)$, for all $t \in I$, for any $\tau \geqslant 0$; the family of operators $\{\sigma_\tau\}_{\tau \geqslant 0}$ forms a semigroup in $\mathcal{C}(I, H_{\mathrm{w}})$ (and a group if $I = \mathbb{R}$).

2.4. The multi-valued evolution operators

Due to the lack of a known global uniqueness result for the three-dimensional Navier–Stokes equations, it is natural to consider the multi-valued evolution operators Σ_t, for $t \geqslant 0$, which associate to each subset E of H, the set $\Sigma_t E$ of all elements of the form $\mathbf{u}(t)$ for some weak solution $\mathbf{u}$ in $\mathcal{U}_{[0,\infty)}$ with initial condition $\mathbf{u}(0)$ in E. In short, this can be written as $\Sigma_t E = \Pi_t(\mathcal{U}_{[0,\infty)} \cap \Pi_0^{-1} E)$. This family of operators satisfies $\Sigma_t \Sigma_s E \subset \Sigma_{t+s} E$, for all $t, s \geqslant 0$, and all sets $E \subset H$. For any given Borel set E in H, it is not known whether $\Sigma_t E$ is Borel, but this set is universally measurable, i.e. it is measurable with respect to the Lebesgue extension of any Borel measure on H.

3. Stationary statistical solutions

In this section we consider the definition and properties of stationary statistical solutions and of time-invariant Vishik–Fursikov measures of the three-dimensional Navier–Stokes equations. We first recall the *cylindrical test functions,* which are the functionals $\Phi : H \to \mathbb{R}$ of the form $\Phi(\mathbf{u}) = \phi((\mathbf{u}, \mathbf{v}_1), \dots, (\mathbf{u}, \mathbf{v}_k))$, where $k \in \mathbb{N}$, ϕ is a C^1 real-valued function on $\mathbb{R}^k$ with compact support, and $\mathbf{v}_1, \dots, \mathbf{v}_k$ belong to V. For such a Φ, we denote by Φ' its Fréchet derivative in H. Then a stationary statistical solution is defined as follows.

Definition 3.1. A stationary statistical solution on H of the three-dimensional Navier–Stokes equations is a Borel probability measure μ on H such that

(i) $\int_H \|\mathbf{u}\|^2 \, \mathrm{d}\mu(\mathbf{u}) < \infty$;
(ii) $\int_H (\mathbf{F}(\mathbf{u}), \Phi'(\mathbf{u})) \, \mathrm{d}\mu(\mathbf{u}) = 0$, for any cylindrical test function Φ;
(iii) For any nonnegative, nondecreasing, continuously-differentiable real-valued function $\psi : [0, \infty) \to \mathbb{R}$ with bounded derivative, we have

$$\int_H \psi'\big(|\mathbf{u}|^2\big)\big(\nu\|\mathbf{u}\|^2 - (\mathbf{f}, \mathbf{u})\big) \, \mathrm{d}\mu(\mathbf{u}) \leqslant 0. \tag{4}$$

Inspired by the approach of Vishik and Fursikov [21], we considered in [3] probability measures carried by the family of all individual weak solutions, which we named Vishik–Fursikov measures; we also defined in [3] the Vishik–Fursikov statistical solutions to be those which are the projections, at each time t, of a Vishik–Fursikov measure. Correspondingly, we define below the Vishik–Fursikov stationary statistical solutions as those which are the projections of Vishik–Fursikov measures which are invariant for the time-translation semigroup defined earlier.

Definition 3.2. Let $I \subset \mathbb{R}$ be an interval unbounded on the right. A Borel probability measure ρ on $\mathcal{C}(I, H_\mathrm{w})$ which is carried by $\mathcal{U}_I^\sharp$ is called a time-invariant Vishik–Fursikov measure over the interval I if it is a Vishik–Fursikov measure (see [3] for the definition) which is invariant with respect to the translation semigroup $\{\sigma_\tau\}_{\tau \geqslant 0}$, in the sense that $\sigma_\tau \rho = \rho$ for all $\tau \geqslant 0$. For such a measure, $\rho_0 = \Pi_t \rho$ is independent of $t \in I$ and is called a Vishik–Fursikov stationary statistical solution.

4. Properties of stationary statistical solutions

4.1. On the carrier of time-invariant Vishik–Fursikov measures

By definition, a time-invariant Vishik–Fursikov measure over an interval I unbounded on the right is carried by $\mathcal{U}_I^\sharp$. But being invariant, it has additional properties as expressed by the following result, which in particular means that it is carried by the set of weak solutions which can be extended to global weak solutions uniformly bounded in H on the whole line:

Theorem 4.1. *Let ρ be a time-invariant Vishik–Fursikov measure over an interval $I \subset \mathbb{R}$ unbounded on the right. If $I = \mathbb{R}$, then ρ is carried by $\mathcal{W}$. If $I = [t_0, \infty)$ or $I = (t_0, \infty)$ for some $t_0 \in \mathbb{R}$, then ρ is carried by $\mathcal{U}_I \cap \Pi_I \mathcal{W}$ and there exists a time-invariant Vishik–Fursikov measure $\tilde{\rho}$ over $\mathbb{R}$ such that $\rho = \Pi_I \tilde{\rho}$, where Π_I is regarded here as the restriction of functions from $\mathbb{R}$ to I.*

4.2. Time-average measures

Using the Kakutani–Riesz Representation Theorem, any generalized limit of time averages of weak solutions generates a stationary statistical solution (we recall that a generalized limit is any continuous linear functional extending the classical limit to the space of essentially bounded measurable functions on the real line, or on any subinterval unbounded on the right; see e.g. [7]). More precisely, given a generalized limit $\mathrm{L{\scriptstyle IM}}_{T\to\infty}$ and a weak solution $\mathbf{u} = \mathbf{u}(t)$ on $t \geqslant t_0$, for some

$t_0 \in \mathbb{R}$, there exists a stationary statistical solution μ on H which is carried by a bounded set $B_H(R)$, for $R > 0$ such that $\mathbf{u}(t) \in B_H(R)$, for all $t \geqslant t_0$, and it is given by the formula

$$\int_H \varphi(\mathbf{v})\,\mathrm{d}\mu(\mathbf{v}) = \underset{T\to\infty}{\mathrm{LIM}} \frac{1}{T}\int_0^T \varphi\big(\mathbf{u}(t_0+t)\big)\,\mathrm{d}t, \quad \forall \varphi \in \mathcal{C}\big(B_H(R)_{\mathrm{w}}\big). \tag{5}$$

Such a solution is called a time-average stationary statistical solution. The next result shows that any such solution is in fact a Vishik–Fursikov stationary statistical solution, so that all time-average measures are time-invariant Vishik–Fursikov measures, and that the projections of time-invariant Vishik–Fursikov measures are Vishik–Fursikov stationary statistical solutions.

Theorem 4.2. *Given a time-average stationary statistical solution μ on H associated with a generalized limit $\mathrm{LIM}_{T\to\infty}$ and a weak solution $\mathbf{u} = \mathbf{u}(t)$, $t \geqslant t_0$, $t_0 \in \mathbb{R}$, there exists a time-invariant Vishik–Fursikov measure ρ over the interval $I = [t_0, \infty)$ for which $\Pi_t\rho = \mu$ for any $t \geqslant t_0$, and*

$$\int_{\mathcal{U}_I^\sharp(R)} \varphi(\mathbf{v})\,\mathrm{d}\rho(\mathbf{v}) = \underset{T\to\infty}{\mathrm{LIM}} \frac{1}{T}\int_0^T \varphi\big(\sigma_t(\mathbf{u})\big)\,\mathrm{d}t, \tag{6}$$

valid for all $\varphi \in \mathcal{C}(\mathcal{C}(I, B_H(R)_{\mathrm{w}}))$. Furthermore, the measure ρ can be extended to a Vishik–Fursikov measure $\tilde{\rho}$ over $\mathbb{R}$, and (6) holds with $\mathcal{U}_I^\sharp$ and ρ replaced by $\mathcal{W}$ and $\tilde{\rho}$, respectively.

4.3. Invariance, accretion and recurrence

Any Vishik–Fursikov stationary statistical solution is associated with a probability measure which is invariant by the time-translation semigroup. Hence, a number of results from ergodic theory applies and yields corresponding results for the Vishik–Fursikov stationary statistical solutions. For general stationary statistical solutions, however, the theory does not apply and some effort is needed to extend, or adapt, some of the results to this context. In that direction a useful concept is that of accretion: a Borel probability measure μ on the phase space H is said to be *accretive* with respect to the family $\{\Sigma_t\}_{t\geqslant 0}$ if $\mu(\Sigma_t E) \geqslant \mu(E)$, for all $t \geqslant 0$ and all Borel subsets E of H.

It was shown in [9] that any time-average stationary statistical solutions are accretive in this sense (see algo [7]). Furthermore, we show in [5] that any Vishik–Fursikov stationary statistical solutions are accretive. It is not known, however, whether all stationary statistical solutions are of Vishik–Fursikov type. For those which might not be of Vishik–Fursikov type but are still accretive the following recurrence-type result is of interest:

Theorem 4.3. *Let μ be an accretive Borel probability measure on H for $\{\Sigma_t\}_{t\geqslant 0}$ and let E be a μ-measurable set with $\mu(E) > 0$. Then, for μ-almost every $\mathbf{u}_0 \in E$, there exists a sequence of positive times $t_n \to \infty$ such that $(\Sigma_{t_n}\mathbf{u}_0) \cap E \neq \emptyset$ for all $n \in \mathbb{N}$. In other words, for μ-almost every $\mathbf{u}_0$ in E, there exists a sequence of positive real numbers $t_n \to \infty$ and a sequence of weak solutions $\mathbf{u}_n$ on $[0,\infty)$ with $\mathbf{u}_n(0) = \mathbf{u}_0$ such that $\mathbf{u}_n(t_n) \in E$.*

Notice that this result is weaker than the classical Poincaré's result since the solution that returns at time t_n may vary with n. This pathology does not occur for Vishik–Fursikov stationary statistical solutions, for which the classical ergodic results can be used, as mentioned above.

5. Connection with the weak global attractor

In the study of the asymptotic behavior of the solutions of the three-dimensional Navier–Stokes equations an important role is played by the weak global attractor, defined as the set $\mathcal{A}_{\mathrm{w}}$ of initial conditions for which there exists a global weak solution uniformly bounded in H on $t \in \mathbb{R}$ [10]; $\mathcal{A}_{\mathrm{w}}$ is the smallest weakly compact set in H which attracts all weak solutions in the weak topology uniformly with respect to initial conditions belonging to any given bounded set in H. It is clear that $\mathcal{A}_{\mathrm{w}} = \Pi_t\mathcal{W}$, for any $t \in \mathbb{R}$. The set $\mathcal{W}$ itself was then shown essentially to be the (trajectory) global attractor of the Navier–Stokes equations in $\mathcal{U}_{\mathbb{R}}$ for the group $\{\sigma_\tau\}_{\tau\in\mathbb{R}}$ (see e.g. Sell and You [17] and Chepyzhov and Vishik [1]). It follows from Theorem 4.1 that any Vishik–Fursikov stationary statistical solution ρ_0 has its support included in $\mathcal{A}_{\mathrm{w}}$, i.e. $\rho_0(\mathcal{A}_{\mathrm{w}}) = 1$.

A partial asymptotic regularity property of the three-dimensional Navier–Stokes equations is expressed by the existence of the "topologically large" subset

$$\begin{aligned}\mathcal{A}_{\mathrm{reg}} = \big\{&\mathbf{u}_0 \in V;\ \exists\delta = \delta(\mathbf{u}_0) > 0 \text{ such that all } \mathbf{u} \in \mathcal{W} \text{ with } \mathbf{u}(0) = \mathbf{u}_0 \\ &\text{are regular on } (-\delta,\delta) \text{ (hence, identical on } (-\delta,\delta))\big\}.\end{aligned}$$

It was shown in [10] that $\mathcal{A}_{\mathrm{reg}}$ is weakly open and weakly dense in $\mathcal{A}_{\mathrm{w}}$.

It is worth noticing that the local uniqueness turns out to follow from the condition that all solutions passing through the same point at the same time are strong solutions. In fact, this regular part of the weak global attractor can also be characterized as

$$\mathcal{A}_{\mathrm{reg}} = \left\{\mathbf{u}_0 \in V;\ \forall \mathbf{u} \in \mathcal{W} \text{ with } \mathbf{u}(0) = \mathbf{u}_0,\ \exists \delta_{\mathbf{u}} > 0 \text{ such that } \mathbf{u} \text{ is a strong solution on } (-\delta_{\mathbf{u}}, \delta_{\mathbf{u}})\right\}.$$

A weakened form of a famous conjecture due to Prodi [15] can be reformulated as follows: *Is* $\mu(\mathcal{A}_{\mathrm{reg}}) = 1$ *for all stationary statistical solutions* μ? A partial result in this direction is the following:

Theorem 5.1. *Consider the following set* $\mathcal{A}'_{\mathrm{reg}}$ *larger than* $\mathcal{A}_{\mathrm{reg}}$:

$$\mathcal{A}'_{\mathrm{reg}} = \left\{\mathbf{u}_0 \in V;\ \exists \mathbf{u} \in \mathcal{W} \textit{ with } \mathbf{u}(0) = \mathbf{u}_0,\ \textit{ and } \delta_{\mathbf{u}} > 0 \textit{ such that } \mathbf{u} \textit{ is a strong solution on } (-\delta_{\mathbf{u}}, \delta_{\mathbf{u}})\right\}.$$

Then $\rho_0(\mathcal{A}'_{\mathrm{reg}}) = 1$ *for all Vishik–Fursikov stationary statistical solution* ρ_0.

In fact, the proof of the statement (i) in the theorem above yields a slightly stronger result:

Theorem 5.2. *Let* ρ *be a time-invariant Vishik–Fursikov measure over* $\mathbb{R}$ *and let* $\mathcal{W}'_{\mathrm{reg}} = \{\mathbf{u} \in \mathcal{W};\ \exists \delta_{\mathbf{u}} > 0$ *such that* $\mathbf{u}$ *is a strong solution on* $(-\delta_{\mathbf{u}}, \delta_{\mathbf{u}})\}$. *Then* $\rho(\mathcal{W}'_{\mathrm{reg}}) = 1$.

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