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Statistics

On the characteristic function of the generalized normal distribution

Sur la fonction caractéristique de la distribution normale généralisée

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ABSTRACT

For the first time, an explicit closed form expression is derived for the characteristic function of the generalized normal distribution (GND). Also derived is an expression for the correlation coefficient between variate-values and their ranks in samples from the GND. The expression for the former involves the Fox–Wright generalized confluent hypergeometric ${}_1\Psi_0$-function, while the latter is expressed *via* the Gaussian hypergeometric ${}_2F_1$.

RÉSUMÉ

Pour la première fois on déduit une expression explicite de la fonction caractéristique de la distribution normale généralisée (GND). On déduit aussi une expression du coefficient de corrélation entre les valeurs d'une variable et leurs rangs dans les échantillonnages de la distribution normale généralisée. La première expression utilise la fonction hypergéométrique confluente de Fox–Wright ${}_1\Psi_0$, la seconde est exprimée via la fonction hypergéométrique gaussienne ${}_2F_1$.

1. Introduction

Consider a r.v. $\xi(\kappa)$ on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ having probability density function (PDF),

$$f_\kappa(\mu,\sigma;x) = \frac{\kappa}{2\sigma\Gamma(1/\kappa)}\exp\left\{-\left|\frac{x-\mu}{\sigma}\right|^\kappa\right\}, \quad \kappa,\sigma>0,\ \mu\in\mathbb{R},\ x\in\mathbb{R}. \tag{1}$$

Then $\xi(\kappa)$ is said to have the *generalized normal distribution* (GND), also known as *power exponential*, *exponential error* or the *generalized Gaussian distribution*. As Kleiber and Kotz reported [1, p. 131], a variation of (1) was first proposed by Subbotin [9], so it is also known as *Subbotin's family of distributions*. We write $\xi(\kappa)\sim \mathrm{GND}(\mu,\sigma,\kappa)$ to denote the fact $\xi(\kappa)$ has the PDF (1). The three parameters μ, σ and κ in (1) represent, respectively, the location, scale and shape of the distribution. Note that $\xi(1)$ is Laplace (or double exponential) and $\xi(2)\sim\mathcal{N}(\mu,\sigma^2/2)$. The pointwise $\lim_{\kappa\to\infty} f_\kappa(\mu,\sigma;x)$ coincides with the PDF of the uniform distribution $\mathcal{U}(\mu-\sigma,\mu+\sigma)$.

Being a generalization of the normal and Laplace distributions, the distribution given by (1) is perhaps the most applicable model in statistics. It has received widespread applications in many applied areas. Some of these areas include: signal processing, material science, quantitative finance, medicine, automation and remote control, telecommunication,

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geophysical research, artificial intelligence, imaging science and photographic technology, information systems, acoustics, software engineering, optics, hardware and architecture, remote sensing, biomedical engineering, radiology, nuclear medicine and medical imaging, transportation science and technology, manufacturing engineering, instruments and instrumentation, physics, analytical chemistry, cybernetics, crystallography, energy and fuels, environmental engineering, computational biology, nanoscience and nanotechnology, neurosciences, operations research and management science, reliability and risk, and dielectrics and electrical insulation.

So, it is important that the properties of (1) are studied comprehensively. A treatment of some mathematical properties of $\mathsf{GND}(\mu, \sigma, \kappa)$ is provided in Nadarajah [4]. The properties studied there include moments, variance, skewness, kurtosis, mean deviation about the mean, mean deviation about the median, Rényi entropy, Shannon entropy, the asymptotic distribution of the extreme order statistics, and estimation procedures by the methods of moments and maximum likelihood.

It appears, however, that the mathematical properties of $\mathsf{GND}(\mu, \sigma, \kappa)$ have not been studied comprehensively. For instance, the characteristic function (CHF) of $\mathsf{GND}(\mu, \sigma, \kappa)$ has not been known in closed form for general κ. Only for the particular cases, $\kappa = 1, 2, 4$ and $\kappa =$ an even positive integer, have some closed forms been derived, see Maturi and Elsayigh [3], Pogány [5]. The CHF is a fundamental tool in probability and statistics. CHFs can also be used as part of procedures for fitting distributions to samples of data. In Theorem 2.1 below, we derive an explicit closed form expression for the CHF of $\mathsf{GND}(\mu, \sigma, \kappa)$ for general κ. This expression involves the Fox–Wright generalized hypergeometric function ${}_p\Psi_q(\cdot)$ with p numerator and q denominator parameters, defined by cf., e.g. Pogány et al. [6, Eq. (9.8)]:

$$
{}_p\Psi_q\left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}, \tag{2}
$$

where the series converges for $A_j, B_k > 0$ and $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$. The Fox–Wright function can be easily evaluated by Maple, Matlab or Mathematica using known in-built routines for hypergeometric functions.

Another property that we study in this Note is the correlation coefficient between variate-values and their ranks in samples. Consider a r.v. ξ on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ with the cumulative distribution function (CDF) $F(x) = \mathsf{P}\{\xi < x\}$. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ denote a random sample of the values of ξ. Let $R_i = R_i(\xi_i)$, $i = \overline{1, n}$ denote the related ranks of ξ_i in the ordered sample $(\xi_{(1)}, \dots, \xi_{(n)})$, $\xi_{(1)} \leqslant \cdots \leqslant \xi_{(n)}$, $\xi_{(j)} \in \boldsymbol{\xi}$. Stuart [7] showed that the correlation coefficient, $\rho(\xi_i, R_i)$, is given by:

$$
\rho(\xi_i, R_i) = \left(\mathsf{E}\xi F(\xi) - \frac{1}{2}\mathsf{E}\xi\right) \cdot \sqrt{\frac{12(n-1)}{(n+1)\mathsf{D}\xi}}. \tag{3}
$$

Here, E and D denote the operations of expectation and variance, respectively.

The correlation coefficient given by (3) has applications in nonparametric statistics, when knowledge of the underlying distribution of the data is very poor. If the correlation is strong then, in practice, one may use the ranks of data instead of the original data. In fact, Stuart [8] showed that for some distributions much information is not lost when the original data are replaced by their corresponding ranks.

Explicit closed form expressions for (3) have been derived by various authors. Stuart [8] considered the case of distributions with no variance. Maturi and Elsayigh [3] considered the case $\xi \sim \mathsf{GND}(\mu, \sigma, 4)$, i.e. with $\kappa = 4$. In Theorem 2.2 below, we consider the general case $\xi \sim \mathsf{GND}(\mu, \sigma, \kappa)$.

2. Main results

Theorem 2.1 derives an explicit closed form expression for the CHF of $\mathsf{GND}(\mu, \sigma, \kappa)$ for general κ.

Theorem 2.1. *Let r.v.* $\xi(\kappa) \sim \mathsf{GND}(\mu, \sigma, \kappa)$, $\kappa > 1$. *The* CHF *of* $\xi(\kappa)$ *can be expressed as*

$$
\varphi_\kappa(t) = \mathsf{E}\exp\{\mathrm{i}t\xi(\kappa)\} = \frac{\sqrt{\pi}\exp\{\mathrm{i}t\mu\}}{\Gamma(1/\kappa)} \cdot {}_1\Psi_1\left[\begin{matrix} (1/\kappa, 2/\kappa) \\ (1/2, 1) \end{matrix}; -\frac{(\sigma t)^2}{4}\right]. \tag{4}
$$

Proof. By direct calculations, we have:

$$
\varphi_\kappa(t) = \int_{\mathbb{R}} \exp\{\mathrm{i}xt\} f_\kappa(\mu, \sigma; x)\,\mathrm{d}x = \frac{\kappa\exp\{\mathrm{i}t\mu\}}{2\Gamma(1/\kappa)} \int_{\mathbb{R}} \exp\{\mathrm{i}x\sigma t\}\exp\{-|x|^\kappa\}\,\mathrm{d}x.
$$

Because $\exp\{-|x|^\kappa\}$ is even we are faced with its cosine-transform, that is

$$
\varphi_\kappa(t) = \frac{\kappa\exp\{\mathrm{i}t\mu\}}{\Gamma(1/\kappa)} \int_0^\infty \cos(x\sigma t)\exp\{-x^\kappa\}\,\mathrm{d}x.
$$

Expanding the cosine-kernel into Maclaurin series and, making use of the legitimate interchange of the sum and the integral, we see,

$$\begin{aligned}\varphi_\kappa(t) &= \frac{\kappa \exp\{\mathrm{i}t\mu\}}{\Gamma(1/\kappa)} \sum_{m=0}^{\infty} \frac{(-1)^m(\sigma t)^{2m}}{(2m)!} \int_0^\infty x^{2m} \exp\{-x^{2k}\}\,\mathrm{d}x \\ &= \frac{\exp\{\mathrm{i}t\mu\}}{\Gamma(1/\kappa)} \sum_{m=0}^{\infty} \frac{(-1)^m(\sigma t)^{2m}}{(2m)!} \int_0^\infty x^{1/\kappa+2m/\kappa-1} \exp\{-x\}\,\mathrm{d}x \\ &= \frac{\exp\{\mathrm{i}t\mu\}}{\Gamma(1/\kappa)} \sum_{m=0}^{\infty} \frac{\Gamma(m/k+1/\kappa)}{\Gamma(2m+1)} \big[-(\sigma t)^2\big]^m. \end{aligned} \tag{5}$$

Applying now the Legendre duplication formula,

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1/2),$$

to the gamma function term $\Gamma(2m+1) = 2m\Gamma(2m)$ in the denominator, we clearly deduce:

$$\varphi_\kappa(t) = \frac{\sqrt{\pi}\exp\{\mathrm{i}t\mu\}}{\Gamma(1/\kappa)} \sum_{m=0}^{\infty} \frac{\Gamma(1/\kappa+2m/\kappa)}{\Gamma(1/2+m)} \frac{[-(\sigma t)^2/4]^m}{m!}.$$

Comparing the last expression with (2), we arrive at (4). The proof is complete, since the resulting Fox–Wright ${}_1\Psi_1$ function converges for $\kappa > 1$. □

Remark 2.1. Following Lunetta [2], Kleiber and Kotz [1, p. 132, Eq. (4.44)] reported a formula for the CHF, $\varphi_\kappa(t)$, equivalent to (5) for the case $\mu = 0$. However, neither Lunetta nor Kleiber and Kotz finalized the explicit formula given by (4). The same formula restricted to $\kappa = 2k \in \mathbb{N}$ has been obtained in Pogány [5, Eq. (5)] by using a slightly different calculation method.

Note that (4) enables one to derive the expectation and the variance of $\xi(\kappa)$ confirming that (see e.g. Nadarajah [4, Eq. (8)]):

$$\mathsf{E}\xi(\kappa) = \mu, \qquad \mathsf{D}\xi(\kappa) = \sigma^2\frac{\Gamma(3/\kappa)}{\Gamma(1/\kappa)}.$$

Now, we state and prove the result on the Stuart's correlation coefficient, which involves the *upper* (*lower*) *incomplete gamma functions*:

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\exp(-t)\,\mathrm{d}t \quad \left(\gamma(s,x) = \int_0^x t^{s-1}\exp(-t)\,\mathrm{d}t\right),$$

connected by the well known relation $\Gamma(s) = \Gamma(s,x) + \gamma(s,x)$.

Theorem 2.2. *Let* $\boldsymbol{\xi}$ *be a sample generated by the r.v.* $\xi(\kappa) \sim \mathsf{GND}(\mu,\sigma,\kappa)$, $\kappa > 3$. *Then we have*:

$$\rho(\xi_i, R_i) = \frac{\sqrt{3}\,\kappa\,\Gamma^{1/2}(3/\kappa)}{\Gamma^{3/2}(1/\kappa)} \cdot {}_2F_1\left[\begin{matrix}1/\kappa, & 3/\kappa \\ 1+1/\kappa\end{matrix}; -1\right] \cdot \sqrt{\frac{n-1}{n+1}}.$$

Proof. It is known that the CDF of $\xi(\kappa) \sim \mathsf{GND}(\mu,\sigma,\kappa)$ is given (Nadarajah [4, Eqs. (5), (6)]) by

$$F(x) = \begin{cases} \dfrac{\Gamma(1/\kappa, [(\mu-x)/\sigma]^\kappa)}{2\Gamma(1/\kappa)}, & x \leqslant \mu, \\ 1 - \dfrac{\Gamma(1/\kappa, [(x-\mu)/\sigma]^\kappa)}{2\Gamma(1/\kappa)}, & x > \mu, \end{cases}$$

which can be rewritten as

$$F(x) = \frac{1}{2}\left[1 + \frac{\operatorname{sgn}(x-\mu)}{\Gamma(1/\kappa)}\gamma\left(1/\kappa, \left|\frac{x-\mu}{\sigma}\right|^\kappa\right)\right].$$

Now we obtain the quantity:

$$\begin{aligned}\mathsf{E}\xi(\kappa)F\big(\xi(\kappa)\big) &= \frac{\kappa}{4\sigma\Gamma(1/\kappa)}\int_{\mathbb{R}} x\left[1+\frac{\operatorname{sgn}(x-\mu)}{\Gamma(1/\kappa)}\gamma\left(1/\kappa, \left|\frac{x-\mu}{\sigma}\right|^{\kappa}\right)\right]\exp\left\{-\left|\frac{x-\mu}{\sigma}\right|^{\kappa}\right\}\mathrm{d}x \\ &= \frac{\kappa\sigma}{4\Gamma(1/\kappa)}\int_{\mathbb{R}}\left(x+\frac{\mu}{\sigma}\right)\left[1+\frac{\operatorname{sgn}(x)}{\Gamma(1/\kappa)}\gamma\big(1/\kappa, |x|^{\kappa}\big)\right]\exp\big\{-|x|^{\kappa}\big\}\,\mathrm{d}x.\end{aligned}$$

Separating the even and odd addends in the integrand, we see,

$$\begin{aligned}\mathsf{E}\xi(\kappa)F\big(\xi(\kappa)\big) &= \frac{\kappa\sigma}{2\Gamma(1/\kappa)}\int_0^{\infty}\left[\frac{\mu}{\sigma}+\frac{x}{\Gamma(1/\kappa)}\cdot\gamma\big(1/\kappa, x^{\kappa}\big)\right]\exp\big\{-x^{\kappa}\big\}\,\mathrm{d}x \\ &= \frac{\mu}{2}+\frac{\sigma}{2\Gamma^2(1/\kappa)}\int_0^{\infty} x^{2/\kappa-1}\gamma(1/\kappa, x)\exp(-x)\,\mathrm{d}x.\end{aligned}$$

Since

$$\gamma(s,x)=\int_0^x t^{s-1}\exp(-t)\,\mathrm{d}t=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^x t^{s+n-1}\,\mathrm{d}t=\frac{x^s}{s}\sum_{m=0}^{\infty}\frac{(s)_m}{(s+1)_m}\cdot\frac{(-x)^m}{m!}=\frac{x^s}{s}\,{}_1F_1\left[\begin{matrix}s\\ s+1\end{matrix};\,-x\right],$$

and

$$\begin{aligned}\int_0^{\infty} x^{a-1}\exp(-bx)\gamma(s;x)\,\mathrm{d}x &= \frac{1}{s}\int_0^{\infty} x^{a+s-1}\exp(-bx)\,{}_1F_1\left[\begin{matrix}s\\ s+1\end{matrix};-x\right]\mathrm{d}x \\ &= \frac{1}{s}\int_0^{\infty} x^{a+s-1}\exp(-bx)\sum_{m=0}^{\infty}\frac{(s)_m}{(s+1)_m}\cdot\frac{(-x)^m}{m!}\,\mathrm{d}x \\ &= \frac{1}{s}\sum_{m=0}^{\infty}\frac{(-1)^m(s)_m}{(s+1)_m m!}\int_0^{\infty} x^{a+s+m-1}\exp(-bx)\,\mathrm{d}x \\ &= \frac{\Gamma(a+s)}{s}\sum_{m=0}^{\infty}\frac{(s)_m(a+s)_m}{(s+1)_m}\frac{(-1/b)^m}{m!} \\ &= \frac{\Gamma(a+s)}{s}\,{}_2F_1\left[\begin{matrix}s, \quad a+s\\ s+1\end{matrix};\,-\frac{1}{b}\right],\end{aligned}$$

taking $a=2/\kappa$, $b=1$ and $s=1/\kappa$, we easily deduce:

$$\mathsf{E}\xi(\kappa)F\big(\xi(\kappa)\big)=\frac{\mu}{2}+\frac{\kappa\sigma\Gamma(3/\kappa)}{2\Gamma^2(1/\kappa)}\,{}_2F_1\left[\begin{matrix}1/\kappa, \quad 3/\kappa\\ 1+1/\kappa\end{matrix};-1\right].$$

Since the hypergeometric function converges (because of $\kappa > 3$), the proof is complete. □

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