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Partial Differential Equations

Identification of two independent coefficients with one observation for the Schrödinger operator in an unbounded strip

Identification de deux coefficients indépendants avec une seule observation pour l'opérateur de Schrödinger dans une bande non bornée

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A R T I C L E I N F O

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A B S T R A C T

This article is devoted to prove a stability result for two independent coefficients (each one depending on only one variable) for a Schrödinger operator in an unbounded strip with only one observation on an unbounded subset of the boundary. For that, we first use the global Carleman estimate proved in Cardoulis et al. (2008) [3]. Then, with a Carleman-type estimate for a first order differential operator (cf. Immanuvilov and Yamamoto (2005) [4]) and an energy estimate, we prove the simultaneous identification of the diffusion coefficient and the potential with only one observation.

R É S U M É

Dans cet article, il s'agit de prouver un résultat de stabilité pour deux coefficients indépendants (chacun dépendant d'une seule variable) pour un opérateur de Schrödinger avec une seule observation sur une partie non bornée du bord. Nous rappelons l'estimation de Carleman globale prouvée dans Cardoulis et al. (2008) [3]. En utilisant une estimation de type Carleman pour un opérateur différentiel du premier ordre (cf. Immanuvilov et Yamamoto (2005) [4]) ainsi qu'une estimation d'énergie, nous obtenons l'identification simultanée du coefficient de diffusion et du potentiel avec une seule observation.

Version française abrégée

Considérons l'équation de Schrödinger (1). L'objet de cette note est d'obtenir un résultat de stabilité pour les deux coefficients indépendants $a = a(x_2)$ et $b = b(x_1)$ avec une seule observation sur une partie non bornée du bord. Nous démontrons d'abord une estimation de Carleman globale avec une seule observation frontière :

Il existe $\lambda_1 \geqslant 1$, $s_1 > 1$ et $C_1 > 0$ tels que l'inégalité de Carleman (4) soit vérifiée pour $s \geqslant s_1$ et $\lambda \geqslant \lambda_1$.

Puis nous donnons une estimation d'énergie (8) et une estimation de type Carleman pour un opérateur différentiel du premier ordre :

Il existe des constantes positives $\lambda_2 > 0$, $s_2 > 0$ et C telles que pour tout $\lambda \geqslant \lambda_2$ et $s \geqslant s_2$ l'inégalité (9) soit vérifiée.

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Nous prouvons ensuite le résultat principal de stabilité Lipschitz pour les deux coefficients a et b avec une seule observation et en supposant connus ces deux coefficients sur le bord :

La norme H^1 du coefficient de diffusion et la norme L^2 du potentiel sont estimées par la norme L^2 de la dérivée en temps de la dérivée normale de la solution sur une partie non bornée du bord (voir (10)).

Nous suivons la méthode développée dans [3] pour établir ce résultat de stabilité.

1. Introduction

This Note is an improvement of the work [3] in the sense that we determine two independent coefficients (each one depending on only one variable) with only one observation.

Several works concern identification of one coefficient for Schrödinger operator in bounded domains (see [1]) or unbounded domains (see [3]). For the simultaneous identification of two coefficients with only one observation, to our knowledge, there is no result.

Let $\Omega = \mathbb{R} \times (0,d)$ be an unbounded strip of $\mathbb{R}^2$ with a fixed width d. Let ν be the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $x = (x_1, x_2)$ and $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{x \in \Gamma;\ x_2 = d\}$ and $\Gamma^- = \{x \in \Gamma;\ x_2 = 0\}$. We consider the following Schrödinger equation:

$$\begin{cases} Hq(x,t) := i\partial_t q(x,t) + \nabla \cdot \big(a(x_2)\nabla q(x,t)\big) + b(x_1)q(x,t) = 0 & \text{in } Q = \Omega \times (0,T), \\ q(x,t) = F(x,t) & \text{on } \Sigma = \partial\Omega \times (0,T), \\ q(x,0) = q_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $a \in \mathcal{C}^3(\overline{\Omega})$, $b \in \mathcal{C}^1(\overline{\Omega})$ and $a(x_2) \geqslant a_{\min} > 0$. Moreover, we assume that a (resp. b) and all its derivatives up to order three (resp. one) are bounded. If we assume that q_0 belongs to $H^4(\Omega)$ and F is sufficiently regular, then (1) admits a solution in $H^1(0,T,H^2(\Omega))$.

Our problem can be stated as follows:

Is it possible to determine the coefficients a and b from the measurement of $\partial_\nu(\partial_t q)$ on Γ^+?

Let q (resp. $\tilde{q}$) be a solution of (1) associated with (a, b, F, q_0) (resp. ($\tilde{a}$, $\tilde{b}$, F, q_0)) satisfying some regularity properties:

Assumption 1.1.

- $\partial_t\tilde{q}$, $\nabla(\partial_t\tilde{q})$ and $\Delta(\partial_t\tilde{q})$ are bounded,
- q_0 is a real valued function in $\mathcal{C}^3(\overline{\Omega})$,
- q_0 and all its derivatives up to order three are bounded.

Our main result is

$$\|a - \tilde{a}\|^2_{H^1(\Omega)} + \|b - \tilde{b}\|^2_{L^2(\Omega)} \leqslant C \big\| \partial_\nu(\partial_t q) - \partial_\nu(\partial_t \tilde{q}) \big\|^2_{L^2((-T,T)\times\Gamma^+)},$$

where C is a positive constant which depends on (Ω, Γ, T) and where the above norms are weighted Sobolev norms.

The major novelty of this Note is to give the simultaneous identification of two coefficients with only one boundary observation in an unbounded domain. Note that we consider here a particular case because each coefficient only depends on one direction. In a work in progress, the general case is studied (see [2]).

We need for our problem a global Carleman estimate and an energy estimate for the operator H given in [3].

We then use a Carleman-type estimate for a first order differential operator proved in [4] for bounded domains and in [3] for unbounded domains.

Using all the previous estimates, we give a stability and uniqueness result for the coefficients a and b.

This Note is organized as follows. In Section 2, we recall an adapted global Carleman estimate for the operator H, an energy estimate and a Carleman-type estimate for a first order differential operator. In Section 3 we prove our main result, a stability result for the coefficients a and b with only one observation.

2. Some useful estimates

2.1. Global Carleman inequality

Let a be a bounded positive function in $\mathcal{C}^3(\overline{\Omega})$ and b be a bounded function in $\mathcal{C}^1(\overline{\Omega})$ such that

Assumption 2.1.

- $a \geqslant a_{\min} > 0$,
- b and all its derivatives up to order one are bounded.

Let $q \in H^1((-T,T); H_0^1(\Omega)) \cap L^2((-T,T); H^2(\Omega))$, $x=(x_1,x_2)$, we denote

$$Hq(x,t) := i\partial_t q(x,t) + \nabla\cdot\big(a(x_2)\nabla q(x,t)\big) + b(x_1)q(x,t).$$

We recall here a global Carleman-type estimate for q with a single observation acting on a part Γ^+ of the boundary Γ in the right-hand side of the estimate (see [3]).

Let $\tilde\beta$ be a $\mathcal{C}^4(\overline{\Omega})$ positive function such that there exist positive constants C_0, C_1, C_{pc} which satisfy

Assumption 2.2.

- $|\nabla\tilde\beta| \geqslant C_0 > 0$ in Ω, $\partial_\nu\tilde\beta \leqslant 0$ on Γ^-,
- $\tilde\beta$ and all its derivatives up to order four are bounded in $\overline{\Omega}$ by C_1,
- $2\Re(D^2\tilde\beta(\zeta,\bar\zeta)) - a\nabla a\cdot\nabla\tilde\beta|\zeta|^2 + 2a^2|\nabla\tilde\beta\cdot\zeta|^2 \geqslant C_{pc}|\zeta|^2$, for all $\zeta\in\mathbb{C}$,

where

$$D^2\tilde\beta = \begin{pmatrix} a\partial_{x_1}(a\partial_{x_1}\tilde\beta) & a\partial_{x_1}(a\partial_{x_2}\tilde\beta) \\ a\partial_{x_2}(a\partial_{x_1}\tilde\beta) & a\partial_{x_2}(a\partial_{x_2}\tilde\beta) \end{pmatrix}.$$

Note that the last assertion of Assumption 2.2 expresses the pseudo-convexity condition for the function β. This assumption imposes restrictive conditions for the choice of the functions a in connection with the function $\tilde\beta$. Similar restrictive conditions have been highlighted for the hyperbolic case in [7,6] and for the Schrödinger operator in [5,3]. Note that there exist functions satisfying such assumptions; indeed, if we consider

$$a(x_2) \in \left\{ f\in C^1(\Omega);\ \exists r_0 \text{ positive constant}, \begin{cases} -f\partial_{x_2}f\partial_{x_2}\tilde\beta \geqslant r_0 > 0, \\ f\partial_{x_2}f\partial_{x_2}\tilde\beta + 2f^2(\partial_{x_2}^2\tilde\beta + (\partial_{x_2}\tilde\beta)^2) \geqslant r_0 > 0 \end{cases} \right\},$$

then the function $\tilde\beta(x_1,x_2)=\tilde\beta(x_2)$ is available (for example, $a(x_2)=e^{-x_2}$ and $\tilde\beta(x_1,x_2)=e^{x_2}$).

We then define $\beta = \tilde\beta + K$ with $K = m\|\tilde\beta\|_\infty$ and $m>1$. For $\lambda>0$ and $t\in(-T,T)$, we define the following weight functions

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{(T+t)(T-t)}, \qquad \eta(x,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(T+t)(T-t)}.$$

We set $\psi = e^{-s\eta}q$, $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ for $s>0$. Let H be the operator defined by

$$Hq := i\partial_t q + \nabla\cdot(a\nabla q) + bq \quad \text{in } \widetilde{Q} = \Omega\times(-T,T), \tag{2}$$

and we introduce the following operators following [1]:

$$\begin{aligned} M_1\psi &:= i\partial_t\psi + \nabla\cdot(a\nabla\psi) + s^2a|\nabla\eta|^2\psi + b\psi, \\ M_2\psi &:= is\partial_t\eta\psi + 2as\nabla\eta\cdot\nabla\psi + s\nabla\cdot(a\nabla\eta)\psi. \end{aligned} \tag{3}$$

Then

$$\int_{-T}^{T}\int_\Omega |M\psi|^2\,dx\,dt = \int_{-T}^{T}\int_\Omega |M_1\psi|^2\,dx\,dt + \int_{-T}^{T}\int_\Omega |M_2\psi|^2\,dx\,dt + 2\Re\left(\int_{-T}^{T}\int_\Omega M_1\psi\overline{M_2\psi}\,dx\,dt\right),$$

where $\bar z$ is the conjugate of z, $\Re(z)$ its real part and $\Im(z)$ its imaginary part. We have to compute the previous scalar product. Then the following result holds:

Theorem 2.3. *Let H, M_1, M_2 be the operators defined respectively by* (2), (3)*. We assume that Assumptions* 2.1 *and* 2.2 *are satisfied. Then there exist $\lambda_0>0$, $s_0>0$ and a positive constant $C=C(\Omega,\Gamma,T)$ such that, for any $\lambda\geqslant\lambda_0$ and any $s\geqslant s_0$, the next inequality holds*:

$$\begin{aligned} &s^3\lambda^4\int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi^3|q|^2\,dx\,dt + s\lambda\int_{-T}^{T}\int_\Omega e^{-2s\eta}\varphi|\nabla q|^2\,dx\,dt + \|M_1(e^{-s\eta}q)\|^2_{L^2(\widetilde{Q})} + \|M_2(e^{-s\eta}q)\|^2_{L^2(\widetilde{Q})} \\ &\quad\leqslant C\left[s\lambda\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi|\partial_\nu q|^2\,\partial_\nu\beta\,d\sigma\,dt + \int_{-T}^{T}\int_\Omega e^{-2s\eta}|Hq|^2\,dx\,dt\right], \end{aligned} \tag{4}$$

for all q satisfying $Hq\in L^2(\Omega\times(-T,T))$, $q\in L^2(-T,T;H_0^1(\Omega))$, $\partial_\nu q\in L^2(-T,T;L^2(\Gamma))$.

2.2. Energy estimate

Let q and $\tilde q$ be solutions of

$$\begin{cases} i\partial_t q + \nabla\cdot(a\nabla q) + bq = 0 & \text{in } \Omega\times(0,T),\\ q(x,t) = F(x,t) & \text{on } \partial\Omega\times(0,T),\\ q(x,0) = q_0(x) & \text{in } \Omega, \end{cases} \qquad \begin{cases} i\partial_t \tilde q + \nabla\cdot(\tilde a\nabla q) + \tilde b\tilde q = 0 & \text{in } \Omega\times(0,T),\\ \tilde q(x,t) = F(x,t) & \text{on } \partial\Omega\times(0,T),\\ \tilde q(x,0) = q_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

where a, b, $\tilde a$ and $\tilde b$ satisfy Assumption 2.1. If we set $u = q - \tilde q$, $v = \partial_t u$, $\alpha = \tilde a - a$ and $\gamma = \tilde b - b$, then v satisfies

$$\begin{cases} i\partial_t v + \nabla\cdot(a\nabla v) + bv = \nabla\cdot\big(\alpha\nabla(\partial_t\tilde q)\big) + \gamma\,\partial_t\tilde q & \text{in } \Omega\times(0,T),\\ v(x,t) = 0 & \text{on } \partial\Omega\times(0,T),\\ v(x,0) = \dfrac{1}{i}\big(\nabla\cdot(\alpha\nabla q_0) + \gamma q_0\big) & \text{in } \Omega. \end{cases} \tag{6}$$

We suppose that Assumption 1.1 is checked, then we extend the function v on $\Omega\times(-T,T)$ by the formula $v(x,t) = -\bar v(x,-t)$ for every $(x,t)\in\Omega\times(-T,0)$. We denote

$$E_1(t) := \int_\Omega e^{-2s\eta(t)}\big|v(t)\big|^2\,dx \quad\text{and}\quad E_2(t) := \int_\Omega a\varphi^{-1}(t)e^{-2s\eta(t)}\big|\nabla v(t)\big|^2\,dx,$$

where $\varphi^{-1} = \frac{1}{\varphi}$. We give an estimate for $E_1(0)$ and $E_2(0)$ in Theorem 2.4.

Theorem 2.4. *Let v be solution of* (6) *in the following class*:

$$v\in C\big([0,T],H^1(\Omega)\big),\qquad \partial_\nu v\in L^2\big(0,T,L^2(\Gamma)\big).$$

We assume that Assumptions 1.1, 2.1 *and* 2.2 *are checked. Then there exists a positive constant $C = C(\Omega,\Gamma,T) > 0$ such that for s and λ sufficiently large*

$$E_1(0) \leqslant C\left(s^{-1/2}\lambda^{-1}\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi\partial_\nu\beta|\partial_\nu v|^2\,d\sigma\,dt + s^{-3/2}\lambda^{-2}\int_\Omega e^{-2s\eta(x,0)}|Hv|^2\right), \tag{7}$$

$$E_2(0) \leqslant C\left(s^2\lambda^2\int_{-T}^{T}\int_{\Gamma^+} e^{-2s\eta}\varphi\partial_\nu\beta|\partial_\nu v|^2\,d\sigma\,dt + s\lambda\int\int_Q e^{-2s\eta}|Hv|^2\right). \tag{8}$$

2.3. Carleman estimate for a first order differential operator

We recall the following Carleman-type estimate for a first order differential operator (see [4] and [3]):

Lemma 2.5. *We consider the first order differential operator P with*

$$Pg = Q_0\cdot\nabla g \quad\textit{and}\quad Q_0 = (q_0, \partial_{x_2}q_0),$$

where Q_0 satisfies $|Q_0\cdot\nabla\beta| \geqslant C > 0$. Then there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ and $C = C(\Omega,\Gamma,T)$ such that for all $\lambda\geqslant\lambda_1$ and $s\geqslant s_1$

$$s^2\lambda^2\int_\Omega \varphi_0 e^{-2s\eta_0}|g|^2\,dx \leqslant C\int_\Omega \varphi_0^{-1}e^{-2s\eta_0}\big|P(g)\big|^2\,dx \tag{9}$$

with $\eta_0(x) := \eta(x,0)$, $\varphi_0(x) := \varphi(x,0)$ and for $g\in H_0^1(\Omega)$.

3. Stability result

Our main stability result is:

Theorem 3.1. *Let q and $\tilde q$ be solutions of* (5) *and $\tilde a - a\in H_0^1(\Omega)$, $\partial_{x_2}(\tilde a - a)\in H_0^1(\Omega)$, $\tilde b - b\in H_0^1(\Omega)$. We assume that there exists a positive constant C such that $|Q_0\cdot\nabla\beta|\geqslant C$ and that Assumptions* 1.1, 2.1, 2.2 *are satisfied. Then there exists a positive constant $C = C(\Omega,\Gamma,T)$ such that for s and λ large enough,*

$$\int_\Omega \varphi_0\,e^{-2s\eta_0}\big(|\tilde a - a|^2 + \big|\nabla(\tilde a - a)\big|^2 + |\tilde b - b|^2\big)\,dx \leqslant C\int_{-T}^{T}\int_{\Gamma^+}\varphi e^{-2s\eta}\partial_\nu\beta\big|\partial_\nu(\partial_t q - \partial_t\tilde q)\big|^2\,d\sigma\,dt. \tag{10}$$

Proof. We apply Lemma 2.5 to the first order partial differential equations satisfied by $\alpha = \tilde{a} - a$ and $\gamma = \tilde{b} - b$ given by the initial condition in (6), the x_1-derivative of this initial condition and also the x_2-derivative. We deduce the following result:

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}|\alpha|^2\,dx \leqslant C \int_\Omega \varphi_0^{-1} e^{-2s\eta_0}\big(\big|v(x,0)\big|^2 + |\gamma|^2\big)\,dx,$$

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}|\gamma|^2\,dx \leqslant C \int_\Omega \varphi_0^{-1} e^{-2s\eta_0}\big(\big|\partial_{x_1} v(x,0)\big|^2 + |\partial_{x_2}\alpha|^2 + |\alpha|^2\big)\,dx,$$

$$s^2\lambda^2 \int_\Omega \varphi_0 e^{-2s\eta_0}|\partial_{x_2}\alpha|^2\,dx \leqslant C \int_\Omega \varphi_0^{-1} e^{-2s\eta_0}\big(\big|\partial_{x_2} v(x,0)\big|^2 + |\alpha|^2 + |\gamma|^2\big)\,dx,$$

for $\alpha, \partial_{x_2}\alpha, \gamma \in H_0^1(\Omega)$.

Then using (7), (8) and for s and λ sufficiently large, we obtain

$$\int_\Omega \varphi_0 e^{-2s\eta_0}\big(|\alpha|^2 + |\partial_{x_2}\alpha|^2 + |\gamma|^2\big)\,dx \leqslant C \int_{-T}^{T}\int_{\Gamma^+} \varphi e^{-2s\eta}\partial_\nu\beta|\partial_\nu v|^2\,d\sigma\,dt. \qquad \square$$

Remark 1. Note that Theorem 3.1 is available with weaker hypotheses on $\tilde{a} - a$ and $\tilde{b} - b$. Indeed if we assume that $\tilde{a} - a \in H^1(\Omega)$, $\tilde{a} - a = \partial_{x_2}(\tilde{a} - a) = \tilde{b} - b = 0$ on $\{x \in \Gamma,\ Q_0 \cdot \nabla\beta\partial_{x_2} Q_0\nu_2 \geqslant 0\}$, and $\tilde{b} - b \in H^1(\Omega)$, then Lemma 2.5 holds and therefore we have (10).

All the previous results are available for bounded domains with weaker hypotheses on $\tilde{q}$ and $\tilde{\beta}$.

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