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On triviality of the Kashiwara–Vergne problem for quadratic Lie algebras

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Abstract

We show that the Kashiwara–Vergne (KV) problem for quadratic Lie algebras (that is, Lie algebras admitting an invariant scalar product) reduces to the problem of representing the Campbell–Hausdorff series in the form $\ln(e^x e^y) = x + y + [x, a(x, y)] + [y, b(x, y)]$, where $a(x, y)$ and $b(x, y)$ are Lie series in x and y. This observation explains the existence of explicit rational solutions of the quadratic KV problem, whereas constructing an explicit rational solution of the full KV problem would probably require the knowledge of a rational Drinfeld associator. It also gives, in the case of quadratic Lie algebras, a direct proof of the Duflo theorem (implied by the KV problem). ***To cite this article: A. Alekseev, C. Torossian, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur le problème de Kashiwara–Vergne pour les algèbres de Lie quadratiques. On montre, dans cette note, que le problème de Kashiwara–Vergne (KV) pour les algèbres de Lie quadratiques se ramène à l'écriture de la formule de Campbell–Hausdorff sous la forme $\ln(e^x e^y) = x + y + [x, a(x, y)] + [y, b(x, y)]$, où $a(x, y)$ et $b(x, y)$ sont des séries de Lie en x et y. Ce résultat explique l'existence dans la littérature, de solutions rationnelles explicites au problème KV quadratique. Notons que la construction d'une solution rationnelle au problème KV général nécessite probablement la connaissance d'un associateur de Drinfeld rationnel. ***Pour citer cet article : A. Alekseev, C. Torossian, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit $\mathbb{K}$ un corps de caractéristique nulle, $\mathfrak{lie}_n$ l'algèbre de Lie libre (complétée) en n générateurs. La série de Campbell–Hausdorff $\mathrm{ch}(x, y) = \ln(e^x e^y)$ est un élément de $\mathfrak{lie}_2$. Plus généralement $\mathrm{ch}(x_1, \dots, x_n) = \ln(e^{x_1} \dots e^{x_n})$ est un élément de $\mathfrak{lie}_n$. Pour $t \in \mathbb{K}^*$, on note $\mathrm{ch}_t(x_1, \dots, x_n) = t^{-1}\,\mathrm{ch}(tx_1, \dots, tx_n)$.

L'algèbre enveloppante universelle de $\mathfrak{lie}_n$, $U(\mathfrak{lie}_n) = \mathrm{Ass}_n$ est l'algèbre (complétée) associative libre en n générateurs. Tout élément α dans Ass_n admet une unique décomposition $\alpha = \alpha_0 + \sum_{i=1}^n (\partial_i \alpha) x_i$ avec $\alpha_0 \in \mathbb{K}$ et $\partial_i \alpha \in \mathrm{Ass}_n$.

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L'action adjointe de $\mathfrak{lie}_n$ s'étend en un homomorphisme d'algèbres $\mathrm{ad}: \mathrm{Ass}_n \to \mathrm{End}(\mathfrak{lie}_n)$. Il est facile de voir que pour α dans $\mathfrak{lie}_n$ on a $\frac{\mathrm{d}}{\mathrm{d}s}\alpha(x_1, \ldots, x_i + sz, \ldots, x_n)|_{s=0} = \mathrm{ad}(\partial_i \alpha)z$. Notons $C_n \subset \mathrm{Ass}_n$ le sous-espace vectoriel engendré par les commutateurs ($ab - ba$ pour $a, b \in \mathrm{Ass}_n$). Soit τ l'unique anti-involution de Ass_n définie par $\tau(\alpha) = -\alpha$ pour $\alpha \in \mathfrak{lie}_n$. Notons $A_n \subset \mathrm{Ass}_n$ le sous-espace propre de τ associé à la valeur propre (-1). Enfin on définit $\mathfrak{tr}_n = \mathrm{Ass}_n / C_n$ le sous-espace gradué des mots cycliques en n lettres et on appelle trace (notée tr) la projection canonique $\mathrm{tr}: \mathrm{Ass}_n \to \mathfrak{tr}_n$. En particulier on a $\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\beta\alpha)$ pour $\alpha, \beta \in \mathrm{Ass}_n$. On définit de manière analogue l'espace des mots cycliques quadratiques $\mathfrak{tr}_n^{\mathrm{quad}} = \mathrm{Ass}_n / \langle A_n, C_n \rangle$ où $\langle A_n, C_n \rangle$ désigne le sous-espace engendré par A_n et C_n et la trace quadratique $\mathrm{tr}^{\mathrm{quad}}: \mathrm{Ass}_n \to \mathfrak{tr}_n^{\mathrm{quad}}$. Remarquons que nos définitions impliquent $\mathrm{tr}^{\mathrm{quad}}(\alpha) = \mathrm{tr}^{\mathrm{quad}}(\tau(\alpha))$ pour tout $\alpha \in \mathrm{Ass}_n$. Lorsque α est dans $\mathfrak{lie}_n$ et k un entier impaire, on en déduit que $\mathrm{tr}^{\mathrm{quad}}(\alpha^k) = 0$.

Le problème de Kashiwara–Vergne (KV) [11] peut être formulé de la manière suivante :

Problème de Kashiwara–Vergne. Trouver une paire de séries de Lie en deux variables $A, B \in \mathfrak{lie}_2$ telle que les équations (1) et (2) soient vérifiées.

Lorsque $\mathfrak{g}$ est une algèbre de Lie de dimension finie sur $\mathbb{K}$, une réponse positive au problème KV fournit une preuve directe à l'isomorphisme de Duflo (isomorphisme explicite entre le centre de l'algèbre enveloppante $Z(U\mathfrak{g})$ et l'anneau des polynômes invariants $(S\mathfrak{g})^{\mathfrak{g}}$) et plus généralement une preuve à l'isomorphisme des algèbres de cohomologie $H(\mathfrak{g}, U\mathfrak{g}) \cong H(\mathfrak{g}, S\mathfrak{g})$ [15,13]. Lorsque $\mathbb{K} = \mathbb{R}$, on étend l'isomorphisme de Duflo au cas des germes de distributions invariantes [5,6].

Lorsque $\mathfrak{g}$ est une algèbre de Lie quadratique, c'est à dire munie d'une forme bilinéaire symétrique non dégénérée (par exemple $\mathfrak{g}$ semi-simple, voir [12] pour d'autres exemples), ces théorèmes résultent d'une version plus faible du problème KV.

Problème Kashiwara–Vergne quadratique. Trouver $A, B \in \mathfrak{lie}_2$ vérifiant les équations (1) et (3).

En effet, lorsque $\mathfrak{g}$ est quadratique on a pour $\alpha \in U\mathfrak{g}$, $\mathrm{tr}_{\mathfrak{g}}\, \mathrm{ad}(\tau(\alpha)) = \mathrm{tr}_{\mathfrak{g}}\, \mathrm{ad}(\alpha)$ avec τ l'anti-involution canonique de $U\mathfrak{g}$. Cette propriété permet de remplacer, au niveau des algèbres de Lie libres, tr par $\mathrm{tr}^{\mathrm{quad}}$. Il est évident que l'équation (1) admet des solutions (et mêmes rationnelles). Le résultat principal de cette note est le théorème suivant qui montre que le problème KV quadratique se résume à la résolution de l'équation (1).

Théorème 0.1. *Toute solution de l'équation* (1) *vérifie l'équation* (3).

1. Introduction

Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathfrak{lie}_n$ be the degree completion of the free Lie algebra with n generators. The Campbell–Hausdorff series $\mathrm{ch}(x, y) = \ln(e^x e^y)$ is an element of $\mathfrak{lie}_2$. Similarly, $\mathrm{ch}(x_1, \ldots, x_n) = \ln(e^{x_1} \ldots e^{x_n})$ is an element of $\mathfrak{lie}_n$. For $t \in \mathbb{K}^*$, let $\mathrm{ch}_t(x_1, \ldots, x_n) = t^{-1}\,\mathrm{ch}(tx_1, \ldots, tx_n)$. Note that $\mathrm{ch}_t(x_1, \ldots, x_n)$ is analytic in t.

The universal enveloping algebra of $\mathfrak{lie}_n$, $U(\mathfrak{lie}_n) = \mathrm{Ass}_n$ is the degree completion of the free associative algebra with n generators. For an element $\alpha \in \mathrm{Ass}_n$ there is a unique decomposition $\alpha = \alpha_0 + \sum_{i=1}^n (\partial_i \alpha) x_i$ with $\alpha_0 \in \mathbb{K}$ and $\partial_i \alpha \in \mathrm{Ass}_n$. Extend the adjoint action of $\mathfrak{lie}_n$ to an algebra homomorphism $\mathrm{ad}: \mathrm{Ass}_n \to \mathrm{End}(\mathfrak{lie}_n)$. It is easy to see that for $\alpha \in \mathfrak{lie}_n$ one has $\frac{\mathrm{d}}{\mathrm{d}s}\alpha(x_1, \ldots, x_i + sz, \ldots, x_n)|_{s=0} = \mathrm{ad}(\partial_i \alpha)z$.

We denote by $C_n \subset \mathrm{Ass}_n$ the subspace spanned by commutators ($ab - ba \in C_n$ for all $a, b \in \mathrm{Ass}_n$). Let τ be the unique anti-involution on Ass_n defined by the property $\tau(\alpha) = -\alpha$ for all $\alpha \in \mathfrak{lie}_n$. Denote by $A_n \subset \mathrm{Ass}_n$ the eigenspace of τ corresponding to the eigenvalue (-1), $\alpha \in A_n$ if $\tau(\alpha) = -\alpha$. Define $\mathfrak{tr}_n = \mathrm{Ass}_n / C_n$ the graded vector space of cyclic words in n letters, and denote by $\mathrm{tr}: \mathrm{Ass}_n \to \mathfrak{tr}_n$ the natural projection. In particular, we have $\mathrm{tr}(\alpha\beta) = \mathrm{tr}(\beta\alpha)$ for all $\alpha, \beta \in \mathrm{Ass}_n$. Similarly, define $\mathfrak{tr}_n^{\mathrm{quad}} = \mathrm{Ass}_n / \langle A_n, C_n \rangle$ and $\mathrm{tr}^{\mathrm{quad}}: \mathrm{Ass}_n \to \mathfrak{tr}_n^{\mathrm{quad}}$ the corresponding projection. Here $\langle A_n, C_n \rangle$ is the subspace of Ass_n spanned by A_n and C_n. The definition implies $\mathrm{tr}^{\mathrm{quad}}(\alpha) = \mathrm{tr}^{\mathrm{quad}}(\tau(\alpha))$ for all $\alpha \in \mathrm{Ass}_n$ and $\mathrm{tr}^{\mathrm{quad}}(\alpha\beta) = \mathrm{tr}^{\mathrm{quad}}(\beta\alpha)$ for all $\alpha, \beta \in \mathrm{Ass}_n$. Note that for all $\alpha \in \mathfrak{lie}_n$ and all k odd we have $\mathrm{tr}^{\mathrm{quad}}(\alpha^k) = \mathrm{tr}^{\mathrm{quad}}(\tau(\alpha^k)) = (-1)^k\,\mathrm{tr}^{\mathrm{quad}}(\alpha^k) = 0$.

Let $\mathfrak{tder}_n$ be the Lie algebra of derivations of $\mathfrak{lie}_n$ with an extra property that for $u \in \mathfrak{tder}_n$ there exist $a_1, \ldots, a_n \in \mathfrak{lie}_n$ such that $u(x_i) = [x_i, a_i]$. Elements of $\mathfrak{tder}_n$ are called tangential derivations. If we assume that a_i does not contain a linear term proportional to x_i, the correspondence between tangential derivations and n-tuples $(a_1, \ldots, a_n)$

is one-to-one. We define simplicial maps $\mathfrak{tder}_n \to \mathfrak{tder}_{n+1}$. For instance, for $u = (A, B) \in \mathfrak{tder}_2$ we introduce $u^{1,2} = (A(x, y), B(x, y), 0)$, $u^{2,3} = (0, A(y, z), B(y, z))$, $u^{12,3} = (A(\mathrm{ch}(x, y), z), A(\mathrm{ch}(x, y), z), B(\mathrm{ch}(x, y), z))$, and $u^{1,23} = (A(x, \mathrm{ch}(y, z)), B(x, \mathrm{ch}(y, z)), B(x, \mathrm{ch}(y, z)))$.

Recall [4, Proposition 3.6] that $\mathrm{div}(u) = \sum_{i=1}^{n} \mathrm{tr}(x_i(\partial_i a_i))$ is a 1-cocycle on $\mathfrak{tder}_n$ with values in $\mathfrak{tr}_n$. Similarly, we define $\mathrm{div}^{\mathrm{quad}}(u) = \sum_{i=1}^{n} \mathrm{tr}^{\mathrm{quad}}(x_i(\partial_i a_i))$. It is a 1-cocycle on $\mathfrak{tder}_n$ with values in $\mathfrak{tr}_n^{\mathrm{quad}}$. It is easy to see that the divergence transforms in a natural way under simplicial maps. For example, for $u \in \mathfrak{tder}_2$ we define $g(x, y) = \mathrm{div}(u) \in \mathfrak{tr}_2$ and we have $\mathrm{div}(u^{1,2}) = g(x, y)$, $\mathrm{div}(u^{2,3}) = g(y, z)$, $\mathrm{div}(u^{12,3}) = g(\mathrm{ch}(x, y), z)$ and $\mathrm{div}(u^{1,23}) = g(x, \mathrm{ch}(y, z))$.

For example, to prove the third equation we use that

$$\mathrm{tr}\big(x\big(\partial_x a\big(\mathrm{ch}(x, y), z\big)\big) + y\big(\partial_y a\big(\mathrm{ch}(x, y), z\big)\big)\big) = \mathrm{tr}\big(\mathrm{ch}(x, y)(\partial_1 a)\big(\mathrm{ch}(x, y), z\big)\big).$$

The same transformation properties under simplicial maps hold for $\mathrm{div}^{\mathrm{quad}}$.

Let TAut_n be the subgroup of automorphisms of $\mathfrak{lie}_n$ with an extra property that for each $g \in \mathrm{TAut}_n$ there exist $b_1, \ldots, b_n \in \mathfrak{lie}_n$ such that $g(x_i) = \exp(\mathrm{ad}_{b_i})x_i$. The group TAut_n is isomorphic to $\mathfrak{tder}_n$ with group multiplication defined by the Campbell–Hausdorff series. Simplicial maps lift to group homomorphisms $\mathrm{TAut}_n \to \mathrm{TAut}_{n+1}$.

2. The Kashiwara–Vergne problem

The (KV) Kashiwara–Vergne problem [11] can be stated in the following way:

Kashiwara–Vergne problem. Find a pair of Lie series in two variables $A, B \in \mathfrak{lie}_2$ such that

$$\big(1 - \exp(-\mathrm{ad}_x)\big)A(x, y) + \big(\exp(\mathrm{ad}_y) - 1\big)B(x, y) = x + y - \mathrm{ch}(y, x), \tag{1}$$

$$\mathrm{tr}\big(x(\partial_x A) + y(\partial_y B)\big) = \frac{1}{2}\,\mathrm{tr}\big(f(x) + f(y) - f\big(\mathrm{ch}(x, y)\big)\big), \tag{2}$$

where $f(x) = x/(e^x - 1) - 1 + x/2 = \sum_{k=2}^{\infty} B_k x^k/k!$ is the generating series of Bernoulli numbers.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$. Then, a positive solution of the KV problem implies the Duflo theorem [10] for $\mathfrak{g}$ (an isomorphism $Z(U\mathfrak{g}) \cong (S\mathfrak{g})^{\mathfrak{g}}$ between the center of the universal enveloping algebra and the ring of invariant polynomials) and the cohomology isomorphism $H(\mathfrak{g}, U\mathfrak{g}) \cong H(\mathfrak{g}, S\mathfrak{g})$ [15,13]. For $\mathbb{K} = \mathbb{R}$, one also obtains the extension of the Duflo isomorphism to germs of invariant distributions [5,6].

Let $\mathfrak{g}$ be a finite dimensional quadratic Lie algebra over $\mathbb{K}$. That is, $\mathfrak{g}$ carries an invariant non-degenerate symmetric bilinear form (*e.g.* $\mathfrak{g}$ is semi-simple, but not necessarily, see [12]). Then, the Duflo theorem (both algebraic and analytic versions) as well as the cohomology isomorphism $H(\mathfrak{g}, U\mathfrak{g}) \cong H(\mathfrak{g}, S\mathfrak{g})$ follow from a weaker version of the KV problem:

Quadratic Kashiwara–Vergne problem. Find a pair of Lie series in two variables $A, B \in \mathfrak{lie}_2$ which verify Eq. (1) and

$$\mathrm{tr}^{\mathrm{quad}}\big(x(\partial_x A) + y(\partial_y B)\big) = \frac{1}{2}\,\mathrm{tr}^{\mathrm{quad}}\big(f(x) + f(y) - f\big(\mathrm{ch}(x, y)\big)\big), \tag{3}$$

with $f(x) = x/(e^x - 1) - 1 + x/2$.

Remark 1. This reduction of the second KV equation from $\mathfrak{tr}_2$ to $\mathfrak{tr}_2^{\mathrm{quad}}$ is related to the following property of traces in the adjoint representation of a quadratic Lie algebra $\mathfrak{g}$. Let $\tau_{\mathfrak{g}}$ be the unique involution of $U\mathfrak{g}$ such that $\tau_{\mathfrak{g}}(\alpha) = -\alpha$ for all $\alpha \in \mathfrak{g}$. Then, for all $\alpha \in U\mathfrak{g}$ we have $\mathrm{tr}_{\mathfrak{g}}\,\mathrm{ad}(\tau(\alpha)) = \mathrm{tr}_{\mathfrak{g}}\,\mathrm{ad}(\alpha)$. At the level of free Lie algebras, this property leads to replacing tr by $\mathrm{tr}^{\mathrm{quad}}$.

It is obvious that Eq. (1) admits many solutions. Indeed, let $a(x, y)$ and $b(x, y)$ be Lie series given by the following formulas

$$a(x, y) = \frac{1 - \exp(-\mathrm{ad}_x)}{\mathrm{ad}_x} A(x, y), \qquad b(x, y) = \frac{\exp(\mathrm{ad}_y) - 1}{\mathrm{ad}_y} B(x, y).$$

Then, Eq. (1) takes the form $[x, a(x, y)] + [y, b(x, y)] = x + y - \mathrm{ch}(y, x)$. The right-hand side is given by a series in Lie monomials of degree greater or equal to two. Since each Lie monomial starts either with a Lie bracket with x or with a Lie bracket with y, we obtain a solution of Eq. (1) for each explicit presentation of the Campbell–Hausdorff formula in terms of Lie monomials (*e.g.* using the Dynkin formula). Furthermore, one can classify solutions of the homogeneous equation $[x, a] + [y, b] = 0$ using Lemma in Section 6 [9] (see below), or using the technique of [8].

The full KV problem admits a solution using the Kontsevich deformation quantization technique [3], and there is another solution [4] using the Drinfeld's theory of associators [9]. At the same time, it is known that the quadratic KV problem is much easier. In particular, it admits explicit rational solutions [16,2] (whereas in the general case, it is plausible that an explicit rational solution of the KV problem amounts to finding an explicit rational associator). There are also two elementary proofs of the Duflo theorem for quadratic Lie algebras: one using Clifford calculus [1], and one using the Kontsevich integral in knot theory [7]. These simplifications in the quadratic case are explained by the following theorem:

Theorem 2.1. *Every solution of Eq.* (1) *verifies Eq.* (3).

To prepare the proof of Theorem 2.1, we recall Lemma in Section 6 [9]. Let $\mathfrak{tr}_n^2$ be the linear span of expressions of the form $\mathrm{tr}(ab)$ for $a, b \in \mathfrak{lie}_n$. The lemma states that there is a one-to-one correspondence between elements $p \in \mathfrak{tr}_n^2$ and n-tuples $a_1, \ldots, a_n \in \mathfrak{lie}_n$ satisfying $\sum_{i=1}^n [x_i, a_i] = 0$. This correspondence is given by formula $\frac{\mathrm{d}}{\mathrm{d}s} p(x_1, \ldots, x_i + sz, \ldots, x_n)|_{s=0} = \mathrm{tr}(za_i)$.

Proposition 2.1. *Let* $a_1, \ldots, a_n \in \mathfrak{lie}_n$ *such that* $\sum_{i=1}^n [x_i, a_i] = 0$, *and let* $u \in \mathfrak{tder}_n$ *be the tangential derivation defined by the* n*-tuple* $(a_1, \ldots, a_n)$. *Then* $\mathrm{div}^{\mathrm{quad}}(u) = \sum_{i=1}^n \mathrm{tr}^{\mathrm{quad}}(x_i(\partial_i a_i)) = 0$.

Proof. Let $p \in \mathfrak{tr}_2^2$ be an element generating the derivation u. Consider

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t} p(x_1, \ldots, x_i + sz_1 + tz_2, \ldots, x_n)|_{s=t=0} &= \frac{\partial}{\partial t} \mathrm{tr}\big(z_1 a_i(x_1, \ldots, x_i + tz_2, \ldots, x_n)\big)\big|_{t=0} \\ &= \mathrm{tr}\big(z_1 \big(\mathrm{ad}(\partial_i a) z_2\big)\big) = \mathrm{tr}\big(\big(\mathrm{ad}\big(\tau(\partial_i a)\big) z_1\big) z_2\big).\end{aligned}$$

Since the left-hand side is symmetric in z_1, z_2, we conclude that $\partial_i a_i = \tau(\partial_i a_i)$. Then,

$$\mathrm{div}^{\mathrm{quad}}(u) = \sum_{i=1}^n \mathrm{tr}^{\mathrm{quad}}\big(x_i(\partial_i a_i)\big) = \sum_{i=1}^n \mathrm{tr}^{\mathrm{quad}}\big(\tau\big(x_i(\partial_i a_i)\big)\big) = -\sum_{i=1}^n \mathrm{tr}^{\mathrm{quad}}\big((\partial_i a_i) x_i\big) = -\mathrm{div}^{\mathrm{quad}}(u),$$

and $\mathrm{div}^{\mathrm{quad}}(u) = 0$, as required. □

Remark 2. Here we presented an algebraic proof of Proposition 2.1 suggested to us by Michele Vergne. One can also give a proof using graphical calculus as in Section 5.1 [14].

The next proposition summarizes known properties of Eq. (1).

Proposition 2.2. *The following three statements are equivalent*:

- $A, B \in \mathfrak{lie}_2$ *is a solution of Eq.* (1).
- *For all* $t \in \mathbb{K}^*$, *the tangential derivation* $u_t \in \mathfrak{tder}_2$ *defined by formula* $u_t(x) = t^{-1}[x, A(tx, ty)], u_t(y) = t^{-1}[y, B(tx, ty)]$ *verifies equation* $u_t(\mathrm{ch}_t(x, y)) = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{ch}_t(x, y)$.
- *The solution* $F_t \in \mathrm{TAut}_2$ *of the differential equation* $F_t^{-1} \frac{\mathrm{d}F_t}{\mathrm{d}t} = u_t$ *with initial condition* $F_0 = 1$ *verifies equation* $F_t(\mathrm{ch}_t(x, y)) = x + y$.

Proof. For equivalence of the first and second statements, see Lemma 3.2 [11]. Equivalence of the second and third statements is obvious, see also Theorem 5.2 [4].

Note that given $F \in \mathrm{TAut}_2$ verifying $F(\mathrm{ch}(x, y)) = x + y$ (as in Proposition 2.2), one can construct $\mathcal{F} \in \mathrm{TAut}_n$ by formula $\mathcal{F} = F^{1,2} F^{12,3} \ldots F^{1\ldots(n-1),n}$ such that

$$F\big(\mathrm{ch}(x_1,\dots,x_n)\big) = F^{1,2}F^{12,3}\dots F^{1\dots(n-1),n}\big(\mathrm{ch}\big(\mathrm{ch}(x_1,x_2,\dots,x_{n-1}),x_n\big)\big)$$
$$= F^{1,2}F^{12,3}\dots F^{1\dots(n-2),n-1}\big(\mathrm{ch}(x_1,\dots,x_{n-1})+x_n\big) = \dots = F^{1,2}\big(\mathrm{ch}(x_1,x_2)+x_3+\dots+x_n\big)$$
$$= x_1+\dots+x_n. \qquad \square$$

Proposition 2.3. *Let $u \in \mathfrak{tder}_n$ such that $u(\mathrm{ch}(x_1,\dots,x_n)) = 0$. Then,* $\mathrm{div}^{\mathrm{quad}}(u) = 0$.

Proof. Define $v = \mathrm{Ad}_{\mathcal{F}}u \in \mathfrak{tder}_n$. We have

$$v(x_1+\dots+x_n) = \mathcal{F}\big(u\big(\mathcal{F}^{-1}(x_1+\dots+x_n)\big)\big) = \mathcal{F}\big(u\big(\mathrm{ch}(x_1,\dots,x_n)\big)\big) = 0.$$

Then $\mathrm{div}^{\mathrm{quad}}(u) = \mathrm{div}^{\mathrm{quad}}(\mathrm{Ad}_{\mathcal{F}^{-1}}(v)) = \mathcal{F}^{-1}\cdot \mathrm{div}^{\mathrm{quad}}(v) = 0$, where we used Proposition 2.1 and the cocycle property of $\mathrm{div}^{\mathrm{quad}}$. $\square$

Proposition 2.4. *Let $A, B \in \mathfrak{lie}_2$ be a solution of Eq.* (1)*, and $u \in \mathfrak{tder}_2$ be the corresponding tangential derivation. Then $U := u^{1,2}+u^{12,3}-u^{1,23}-u^{2,3}$ verifies $U(\mathrm{ch}(x,y,z)) = 0$.*

Proof. Consider $\mathrm{ch}_t(x,y,z) = \mathrm{ch}_t(\mathrm{ch}_t(x,y),z)$. We have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{ch}_t(x,y,z)|_{t=1} = \left(\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{ch}_p\big(\mathrm{ch}_q(x,y),z\big) + \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{ch}_p\big(\mathrm{ch}_q(x,y),z\big)\right)_{p=q=1}$$
$$= u^{12,3}\big(\mathrm{ch}(x,y,z)\big) + u^{1,2}\big(\mathrm{ch}(x,y,z)\big).$$

Similarly, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{ch}_t(x,y,z)|_{t=1} = \left(\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{ch}_p\big(x,\mathrm{ch}_q(y,z)\big) + \frac{\mathrm{d}}{\mathrm{d}q}\mathrm{ch}_p\big(x,\mathrm{ch}_q(y,z)\big)\right)_{p=q=1}$$
$$= u^{1,23}\big(\mathrm{ch}(x,y,z)\big) + u^{2,3}\big(\mathrm{ch}(x,y,z)\big).$$

By combining these two equations we arrive at $U(\mathrm{ch}(x,y,z)) = 0$. $\square$

Proposition 2.5. *Let $A, B \in \mathfrak{lie}_2$ be a solution of Eq.* (1)*, $u \in \mathfrak{tder}_2$ be the corresponding tangential derivation, and $g = \mathrm{div}^{\mathrm{quad}}(u) \in \mathrm{tr}_2^{\mathrm{quad}}$. Then, there is $h \in \mathrm{tr}_1^{\mathrm{quad}}$ such that $g(x,y) = h(x)+h(y)-h(\mathrm{ch}(x,y))$.*

Proof. Propositions 2.3 and 2.4 imply that for $U \in \mathfrak{tder}_3$ defined in Proposition 2.4 we have $\mathrm{div}^{\mathrm{quad}}(U) = 0$. That is,

$$0 = \mathrm{div}^{\mathrm{quad}}(U) = \mathrm{div}^{\mathrm{quad}}\big(u^{1,2}\big) + \mathrm{div}^{\mathrm{quad}}\big(u^{12,3}\big) - \mathrm{div}^{\mathrm{quad}}\big(u^{1,23}\big) - \mathrm{div}^{\mathrm{quad}}\big(u^{2,3}\big)$$
$$= g(x,y) + g\big(\mathrm{ch}(x,y),z\big) - g\big(x,\mathrm{ch}(y,z)\big) - g(y,z). \tag{4}$$

Replacing tr by $\mathrm{tr}^{\mathrm{quad}}$ in Theorem 2.1 and Proposition 2.2 of [4] yields $g(x,y) = h(x)+h(y)-h(\mathrm{ch}(x,y))$ for some $h \in \mathrm{tr}_1^{\mathrm{quad}}$. $\square$

We conclude the proof of the main result of this paper with the following proposition.

Proposition 2.6. *Let $A, B \in \mathfrak{lie}_2$ be a solution of* (1)*, $u \in \mathfrak{tder}_2$ be the corresponding tangential derivation, and assume* $\mathrm{div}^{\mathrm{quad}}(u) = h(x)+h(y)-h(\mathrm{ch}(x,y))$ *for some $h \in \mathrm{tr}_1^{\mathrm{quad}}$. Then, $h(x) = \mathrm{tr}^{\mathrm{quad}} f(x)$ for $f = x/(e^x-1)-1+x/2$.*

Proof. Let $\tilde{f}$ be a formal power series in one variable such that $h(x) = \mathrm{tr}^{\mathrm{quad}}\tilde{f}(x)$. Proposition 6.1 in [4] implies that $\tilde{f}_{\mathrm{even}}(x) = f(x)$, where $\tilde{f}_{\mathrm{even}}(x) = (\tilde{f}(x)+\tilde{f}(-x))/2$. Since $\mathrm{tr}^{\mathrm{quad}}(x^n) = 0$ for n odd, we conclude $h(x) = \mathrm{tr}^{\mathrm{quad}}\tilde{f}(x) = \mathrm{tr}^{\mathrm{quad}} f(x)$, as required. $\square$

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