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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1243–1248

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations/Mathematical Physics

Scattering by a Minkowski brane world

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Received 19 January 2009; accepted after revision 2 September 2009

Available online 3 October 2009

Presented by Jean-Michel Bony

Abstract

We study the wave equation for the gravitational waves in the Randall–Sundrum brane cosmology model. The global Cauchy problem is well posed in the functional framework associated with the energy. The solutions are the sum of a free massless wave propagating on the brane (the *graviton*), and a superposition of massive Klein–Gordon waves (the *Kaluza–Klein tower*). We compute the kernel of the truncated resolvent in term of Hankel functions. We develop the complete asymptotic analysis of the Kaluza–Klein towers: L^1–L^∞ and L^2–L^∞ estimates, global L^p Strichartz estimates, existence and asymptotic completeness of the wave operators, computation of the scattering matrix, determination of the resonances on the logarithmic Riemann surface. ***To cite this article: A. Bachelot, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Diffusion par un univers-brane de Minkowski. Nous étudions l'équation des ondes gravitationnelles dans le modèle de cosmologie branaire de Randall–Sundrum. Le problème de Cauchy global est bien posé dans l'espace des champs d'énergie finie. Les solutions se décomposent de façon unique en la somme d'une onde libre sans masse se propageant sur la brane de Minkowski (le *graviton*) et d'un somme continue de champs massifs de Klein–Gordon (la *tour de Kaluza–Klein*). Le résolvant tronqué est explicitement exprimé à l'aide de fonctions de Hankel. Nous faisons l'analyse asymptotique complète des tours de Kaluza–Klein : estimations L^1–L^∞, L^2–L^∞, estimations globales L^p de type Strichartz, existence et complétude des opérateurs d'ondes, calcul de la matrice de diffusion, détermination des résonances sur la surface de Riemann du logarithme. ***Pour citer cet article : A. Bachelot, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Le modèle de cosmologie branaire de Randall et Sundrum [4], décrit notre monde plat, comme une membrane de Minkowski $\{z=0\}$ immergée dans la variété lorentzienne non régulière

$$\mathcal{M}=\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times\mathbb{R}_z,\qquad \mathrm{d}s^2=\big(1+|z|\big)^{-2}\big(\mathrm{d}t^2-\mathrm{d}\mathbf{x}^2-\mathrm{d}z^2\big).$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.09.004

Les fluctuations gravitationnelles sont solutions d'une équation des ondes comportant un potentiel singulier localisé sur la membrane :

$$\partial_t^2\Phi+\mathbf{H}\Phi=0,\quad \mathbf{H}:=-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4}\left(\frac{1}{1+|z|}\right)^2-3\delta_0(z),$$

et leur énergie conservée est définie par :

$$\|\partial_t\Phi\|^2_{L^2(\mathbb{R}^4_{\mathbf{x},z})}+\left\|\mathbf{H}^{\frac{1}{2}}\Phi\right\|^2_{L^2(\mathbb{R}^4_{\mathbf{x},z})}=\|\nabla_{t,\mathbf{x}}\Phi\|^2_{L^2(\mathbb{R}^4_{\mathbf{x},z})}+\left\|\partial_z\Phi+\frac{3}{2}\frac{z}{|z|}\left(\frac{1}{1+|z|}\right)\Phi\right\|^2_{L^2(\mathbb{R}^4_{\mathbf{x},z})}.$$

Le problème de Cauchy global est bien posé dans le cadre fonctionnel associé. Toute solution d'énergie finie s'exprime comme la somme d'une solution particulière du type

$$\Phi_{grav}(t,\mathbf{x},z):=\phi_0(t,\mathbf{x})\big(1+|z|\big)^{-\frac{3}{2}},\quad \partial_t^2\phi_0-\Delta_{\mathbf{x}}\phi_0=0,$$

appelée *graviton*, et d'une autre solution particulière, la *tour de Kaluza–Klein*, de la forme

$$\Phi_{KK}(t,\mathbf{x},z)=\sum_{\pm}\int_0^\infty\phi_m^\pm(t,\mathbf{x})f_m^\pm(z)\,\mathrm{d}m,\quad \partial_t^2\phi_m^\pm-\Delta_{\mathbf{x}}\phi_m^\pm+m^2\phi_m^\pm=0,$$

avec

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{15}{4}\left(\frac{1}{1+|z|}\right)^2-3\delta_0(z)\right]f_m^\pm=m^2f_m^\pm,$$
$$f_m^\pm(-z)=\pm f_m^\pm(z),\qquad f_m^\pm(z)=O(\sqrt{m}),\quad m\to 0,$$
$$f_m^+(z)\sim-\sqrt{\frac{2}{\pi}}\cos(mz),\qquad f_m^-(z)\sim\sqrt{\frac{2}{\pi}}\sin(mz),\quad m\to\infty.$$

La résolvante tronquée $\mathbf{1}_{[0,R_1]}(|\mathbf{x}|+|z|)(\mathbf{H}-\lambda^2)^{-1}\mathbf{1}_{[0,R]}(|\nabla_{\mathbf{x}}|)\mathbf{1}_{[0,R_2]}(|\mathbf{x}|+|z|)$ est exprimable à l'aide d'intégrales de fonctions de Hankel, $H_\nu^{(j)}$, $j,\nu=1,2$, et, considéré comme une fonction de λ à valeur dans $\mathcal{L}(L^2(\mathbb{R}^4_{\mathbf{x},z}))$, il admet un prolongement analytique dans le complémentaire, dans la surface de Riemann du logarithme $\widetilde{\mathbb{C}^*}$, d'un réseau d'hyperboles : $\{\lambda\in\widetilde{\mathbb{C}^*};\ H_\nu^{(1)}(\sqrt{\lambda^2-r^2})\neq 0\ \forall r\in[0,R],\nu=1,2\}$.

Les propriétés du graviton étant trivialement celles des ondes libres dans $\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}$, on fait une étude complète du comportement asymptotique des tours de Kaluza–Klein, qui différencient l'univers de Minkowski de celui de Randall–Sundrum. Au voisinage de la membrane, des estimations L^1–L^∞ et L^2–L^∞ montrent que Φ_{KK} décroît comme $|t|^{-\frac{3}{2}}$ dans $L^\infty(\mathbb{R}^3_{\mathbf{x}}\times[-R,R]_z)$. Par ailleurs Φ_{KK} satisfait des inégalités de type Strichartz dans $L^{\frac{10}{3}}(\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times[-R,R]_z)$ et $L^\infty([-R,R]_z;L^4(\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}))$. On compare aussi Φ_{KK} dans tout l'espace avec les ondes libres de $\mathbb{R}^{1+4}$ solutions de $(\partial_t^2-\Delta_{\mathbf{x}}-\partial_z^2)\Phi_\infty^\pm=0$, $(t,\mathbf{x},z)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}$. On établit l'existence et la complétude asymptotique des opérateurs d'onde définis par les limites :

$$\lim_{t\to\pm\infty}\left\|\nabla_{\mathbf{x}}\Phi(t)-\nabla_{\mathbf{x}}\Phi_\infty^\pm(t)\right\|_{L^2(\mathbb{R}^4_{\mathbf{x},z})}+\left\|\partial_z\Phi(t)+\frac{3}{2}\frac{z}{|z|}\left(\frac{1}{1+|z|}\right)\Phi(t)-\partial_z\Phi_\infty^\pm(t)\right\|^2_{L^2(\mathbb{R}^4_{\mathbf{x},z})}=0.$$

L'opérateur de diffusion $\Phi_-\mapsto\Phi_+$ est une isométrie sur l'espace des ondes d'énergie finies dans $\mathbb{R}^{1+4}$, et sa représentation spectrale (la "matrice de scattering") admet un prolongement analytique sur la surface du logarithme privée du réseau de demi-droites radiales

$$\big\{\lambda\in\widetilde{\mathbb{C}^*},\ H_\nu^{(1)}(\alpha\lambda)H_\nu^{(2)}(-\alpha\lambda)\neq 0,\ \forall\alpha\in]0,1],\ \nu=1,2\big\}.$$

Ces résultats fondent les assertions physiques : le terme dominant des fluctuations gravitationnelles est le graviton sans masse localisé près de la membrane ; la tour de Kaluza–Klein est une superposition de champs massifs qui est dispersive dans $\mathbb{R}^5$ et décroît plus rapidement ; les résonances de diffraction par la membrane de Minkowski forment un continuum. L'ensemble de ces propriétés asymptotiques suggère que ce modèle de cosmologie branaire est linéairement stable. La véritable stabilité non-linéaire de la membrane de Minkowski est un problème ouvert aussi fondamental que difficile.

1. Dynamics and brane scattering in the brane cosmology model

In the famous brane cosmology model due to L. Randall and R. Sundrum [4], our flat world is described by a Minkowski brane $z=0$, imbedded in the five-dimensional Lorentzian manifold

$$\mathcal{M}=\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times\mathbb{R}_z,\qquad \mathrm{d}s^2=\big(1+|z|\big)^{-2}\big(\mathrm{d}t^2-\mathrm{d}\mathbf{x}^2-\mathrm{d}z^2\big).$$

The submanifolds $z\gtrless 0$ are pieces of the Anti-De Sitter universe, but unlike this one, $\mathcal{M}$ is globally hyperbolic. The gravitational fluctuations around this background obey to the master equation:

$$\partial_t^2\Phi+\mathbf{H}\Phi=0,\quad \mathbf{H}:=-\Delta_{\mathbf{x}}-\partial_z^2+\frac{15}{4}\left(\frac{1}{1+|z|}\right)^2-3\delta_0(z) \tag{1}$$

where the Dirac distribution denotes the simple layer on the brane. Our work is devoted to a complete analysis of this equation. In this Note, we present the main results, the detailed proofs are given in [1].

We show that $\mathbf{H}$ endowed with the domain $\mathfrak{D}(\mathbf{H}):=\{u\in H^1(\mathbb{R}^3_{\mathbf{x}}\times\mathbb{R}_z);\,\mathbf{H}u\in L^2(\mathbb{R}^3_{\mathbf{x}}\times\mathbb{R}_z)\}$, is a positive self-adjoint operator on $L^2(\mathbb{R}^4)$ and 0 is not an eigenvalue. The domain of $\mathbf{H}^{\frac12}$ is $H^1(\mathbb{R}^4)$ and

$$\big\|\mathbf{H}^{\frac12}u\big\|^2_{L^2(\mathbb{R}^4)}=\|\nabla_{\mathbf{x}}u\|^2_{L^2(\mathbb{R}^4)}+\left\|\partial_z u+\frac32\frac{z}{|z|}\left(\frac{1}{1+|z|}\right)u\right\|^2_{L^2(\mathbb{R}^4)}.$$

We introduce the Hilbert space $\mathfrak{W}^1(\mathbb{R}^4)$ that is the closure of $H^1(\mathbb{R}^4)$ for this norm. Then for any $\Phi_0\in\mathfrak{W}^1(\mathbb{R}^4)$, $\Phi_1\in L^2(\mathbb{R}^4)$, there exists a unique solution Φ of (1), such that

$$\Phi\in C^0\big(\mathbb{R}_t;\mathfrak{W}^1\big),\quad \partial_t\Phi\in C^0\big(\mathbb{R}_t;L^2\big(\mathbb{R}^4\big)\big),\qquad \Phi(0,\mathbf{x},z)=\Phi_0(\mathbf{x},z),\quad \partial_t\Phi(0,\mathbf{x},z)=\Phi_1(\mathbf{x},z). \tag{2}$$

These waves are called finite energy solutions, and satisfy the conservation law:

$$\forall t\in\mathbb{R},\quad \big\|\Phi(t)\big\|^2_{\mathfrak{W}^1(\mathbb{R}^4)}+\big\|\partial_t\Phi(t)\big\|^2_{L^2(\mathbb{R}^4)}=\|\Phi_0\|^2_{\mathfrak{W}^1(\mathbb{R}^4)}+\|\Phi_1\|^2_{L^2(\mathbb{R}^4)}.$$

We can very simply perform such a solution of finite energy. We denote $BL^1(\mathbb{R}^N)$ the Beppo Levi space on $\mathbb{R}^N$, defined as the closure of $C_0^\infty(\mathbb{R}^N)$ for the norm $\|\nabla u\|_{L^2}$. Given $\varphi_0\in BL^1(\mathbb{R}^3_{\mathbf{x}})$, $\varphi_1\in L^2(\mathbb{R}^3_{\mathbf{x}})$, we consider $\phi_0\in C^0(\mathbb{R}_t;BL^1(\mathbb{R}^3_{\mathbf{x}}))$, with $\partial_t\phi_0\in C^0(\mathbb{R}_t;L^2(\mathbb{R}^3_{\mathbf{x}}))$, the solution of the wave equation on the Minkowski brane, $\partial_t^2\phi_0-\Delta_{\mathbf{x}}\phi_0=0$, with initial data $\phi_0(0,\mathbf{x})=\varphi_0(\mathbf{x})$, $\partial_t\phi_0(0,\mathbf{x})=\varphi_1(\mathbf{x})$. Then

$$\Phi(t,\mathbf{x},z)=\phi_0(t,\mathbf{x})f_0(z),\quad f_0(z):=\big(1+|z|\big)^{-\frac32},$$

is a solution of (1). We call *massless graviton* any solution with initial data $\Phi_0\in BL^1(\mathbb{R}^3_{\mathbf{x}})\otimes\mathbb{C}f_0(z)$, $\Phi_1\in L^2(\mathbb{R}^3_{\mathbf{x}})\otimes\mathbb{C}f_0(z)$. We note that its energy remains obviously localized near the brane on which it is propagating. We are mainly interested in the *Kaluza–Klein towers* that are solutions with data in

$$\mathfrak{K}^1:=\big(BL^1\big(\mathbb{R}^3_{\mathbf{x}}\big)\otimes\mathbb{C}f_0(z)\big)^{\perp_{\mathfrak{W}^1}},\qquad \mathfrak{K}^0:=\big(L^2\big(\mathbb{R}^3_{\mathbf{x}}\big)\otimes\mathbb{C}f_0(z)\big)^{\perp_{L^2}}.$$

The crucial point is that these waves are superpositions of massive Klein–Gordon fields, propagating on the brane. This result is obtained by constructing the distorted Fourier transforms for the self-adjoint operator on $L^2(\mathbb{R}_z)$, $\mathbf{h}:=-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{15}{4}(\frac{1}{1+|z|})^2-3\delta_0(z)$, $\mathfrak{D}(\mathbf{h}):=\{u\in H^1(\mathbb{R}_z);\,\mathbf{h}u\in L^2(\mathbb{R}_z)\}$.

Theorem 1.1. *There exists $f_m^\pm(z)\in C^0([0,\infty[_m\times\mathbb{R}_z)$ with $f_m^\pm(z)=O(\sqrt{m})$, $m\to0$, $f_m^+(z)\sim-\sqrt{\frac{2}{\pi}}\cos(mz)$, $f_m^-(z)\sim\sqrt{\frac{2}{\pi}}\sin(mz)$, $m\to\infty$, such that for any $\Phi_0\in\mathfrak{K}^1$, $\Phi_1\in\mathfrak{K}^0$, the solution Φ of* (1) *and* (2) *can be expressed as*

$$\Phi(t,\mathbf{x},z)=\sum_{\pm}\lim_{M\to\infty}\int_0^M\phi_m^\pm(t,\mathbf{x})f_m^\pm(z)\,\mathrm{d}m\quad \text{in } C^0\big(\mathbb{R}_t;\mathfrak{K}^1\big), \tag{3}$$

where for almost all $m>0$, $\phi_m\in C^0(\mathbb{R}_t;H^1(\mathbb{R}^3_{\mathbf{x}}))\cap C^1(\mathbb{R}_t;L^2(\mathbb{R}^3_{\mathbf{x}}))$ is solution of $\partial_t^2\phi_m^\pm-\Delta_{\mathbf{x}}\phi_m^\pm+m^2\phi_m^\pm=0$. Moreover,

$$\|\Phi_0\|^2_{\mathfrak{W}^1(\mathbb{R}^4)} + \|\Phi_1\|^2_{L^2(\mathbb{R}^4)} = 2\sum_{\pm}\int_0^\infty \|\nabla_{t,\mathbf{x}}\phi_m^{\pm}(t)\|^2_{L^2(\mathbb{R}^3)} + m^2\|\phi_m^{\pm}(t)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}m.$$

Since $\mathbf{H}$ is a positive self-adjoint operator, $(\mathbf{H}-\lambda^2)^{-1}$ is well defined in $\mathcal{L}(L^2(\mathbb{R}^4))$ for all $\lambda \in \mathbb{C}^*$ with $0 < \arg\lambda < \pi$. If we add a cut-off in energy relatively to the brane, i.e. we consider $(\mathbf{H}-\lambda^2)^{-1}\mathbf{1}_{[0,R]}(|\nabla_{\mathbf{x}}|)$, we can express the kernel explicitly by using the Hankel functions that are holomorphic on the whole Riemann surface of the logarithm $\widetilde{\mathbb{C}^*}$. We introduce

$$\begin{aligned}\mathbf{K}(m;z,z') := \frac{\pi}{4i}\sqrt{(1+z)(1+z')}\big[&H_2^{(2)}(m)H_2^{(1)}\big(m(1+z\wedge z')\big)\\ &- H_2^{(1)}(m)H_2^{(2)}\big(m(1+z\wedge z')\big)\big]H_2^{(1)}\big(m(1+z\vee z')\big)\end{aligned}$$

where $z\wedge z' := \min(z,z')$, $z\vee z' := \max(z,z')$. For $x \in \mathbb{C}^*$, we denote $\sqrt{x}$ the principal branch of the square root, i.e. $0 \leqslant \arg\sqrt{x} < \pi$ when $0 \leqslant \arg\lambda < 2\pi$. For $R \in [0,\infty]$, we put

$$\Sigma_R := \big\{\lambda \in \widetilde{\mathbb{C}^*};\ \exists r \in [0,R],\ H_1^{(1)}\big(\sqrt{\lambda^2-r^2}\big)H_2^{(1)}\big(\sqrt{\lambda^2-r^2}\big) = 0\big\}.$$

Σ_∞ is the set of the *Brane Resonances* of the physicists (see [5]).

Theorem 1.2. *For any $R > 0$, $\lambda \in \mathbb{C}^*$, $0 < \arg\lambda < \pi$, $F \in L^1 \cap L^2(\mathbb{R}^4)$, we have*

$$\big(\mathbf{H}-\lambda^2\big)^{-1}\mathbf{1}_{[0,R]}\big(|\nabla_{\mathbf{x}}|\big)F(\mathbf{x},z) = \int_{\mathbb{R}^4}\mathbf{K}_R(\lambda;\mathbf{x},z;\mathbf{x}',z')F(\mathbf{x}',z')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}z',$$

where this integral converges absolutely and the kernel of the truncated resolvent is given by

$$\begin{aligned}&\mathbf{K}_R(\lambda;\mathbf{x},z;\mathbf{x}',z')\\ &\quad := \int_0^R \frac{\sin(r|\mathbf{x}-\mathbf{x}'|)}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\left[\frac{zz'}{|zz'|}\frac{1}{H_2^{(1)}(\sqrt{\lambda^2-r^2})} - \frac{1}{H_1^{(1)}(\sqrt{\lambda^2-r^2})}\right]\mathbf{K}\big(\sqrt{\lambda^2-r^2};|z|,|z'|\big)r\,\mathrm{d}r.\end{aligned}$$

For any $R_j > 0$, the truncated resolvent

$$\mathbf{1}_{[0,R_1]}\big(|\mathbf{x}|+|z|\big)\big(\mathbf{H}-\lambda^2\big)^{-1}\mathbf{1}_{[0,R_0]}\big(|\nabla_{\mathbf{x}}|\big)\mathbf{1}_{[0,R_2]}\big(|\mathbf{x}|+|z|\big)$$

considered as a $\mathcal{L}(L^2(\mathbb{R}^4_{\mathbf{x},z}))$-valued function of λ, has an analytic continuation on $\widetilde{\mathbb{C}^*}\setminus\Sigma_{R_0}$.

The well known L^p-estimates due to von Wahl [7], assure that for suitable smooth initial data, the solutions of the D'Alembertian and the massive Klein–Gordon equation on the Minkowski space–time $\mathbb{R}^{1+N}$, decay respectively as $|t|^{-\frac{N-1}{2}}$ and $|t|^{-\frac{N}{2}}$. Therefore the massless graviton decays as $|t|^{-1}$. The fact that is meaningful of the physical point of view, is that the Kaluza–Klein tower is more dispersive: it decays as $|t|^{-\frac{3}{2}}$, like the free waves in $\mathbb{R}^{1+4}$. The key of this phenomenon is the representation (3) of the field as a sum of massive Klein–Gordon fields in $\mathbb{R}^{1+3}$, where $f_m = O(\sqrt{m})$. Thanks to this asymptotic behaviour near zero, the Kaluza–Klein tower decays as the Klein–Gordon fields despite the fact that the integral in (3) expends up to $m = 0$. In order to establish the L^1–L^∞ and L^2–L^∞ estimates near the brane, we introduce several norms inspired by [2]:

$$\|\Phi_j\|_{X_{j,\varepsilon}} := \frac{1}{\varepsilon}\sum_{|\alpha|+j\leqslant 3}\big\|\big(1+|z|\big)^{\varepsilon}\partial_{\mathbf{x}}^{\alpha}\Phi_j\big\|_{L^1(\mathbb{R}^4)} + \sum_{|\alpha|+j\leqslant 3}\ \sum_{l=1}^{4-|\alpha|-j}\big\|\partial_z^l\partial_{\mathbf{x}}^{\alpha}\Phi_j\big\|_{L^1(\mathbb{R}^3_{\mathbf{x}}\times\mathbb{R}^*_z)},$$

$$\begin{aligned}\|\Phi_j\|_{Y_{j,\varepsilon}} := &\sum_{|\alpha|+j\leqslant 2}\ \sum_{0\leqslant p,q}\big\|\chi_p\big(|\mathbf{y}|\big)\chi_q\big(|z|\big)\big(1+|\mathbf{y}|\big)^{\frac{3}{2}}\big(1+|z|\big)^{\frac{1}{2}}\partial_{\mathbf{y}}^{\alpha}\Phi_j\big\|_{L^2(\mathbb{R}^3_{\mathbf{y}}\times\mathbb{R}_z)}\\ &+\frac{1}{\varepsilon}\sum_{|\alpha|+j=3}\ \sum_{0\leqslant p,q}\big\|\chi_p\big(|\mathbf{y}|\big)\chi_q\big(|z|\big)\big(1+|\mathbf{y}|\big)^{\frac{3}{2}}\big(1+|z|\big)^{\frac{1}{2}+\varepsilon}\partial_{\mathbf{y}}^{\alpha}\Phi_j\big\|_{L^2(\mathbb{R}^3_{\mathbf{y}}\times\mathbb{R}_z)}\end{aligned}$$

$$+\sum_{l=1}^{4}\sum_{|\alpha|+j\leqslant 4-l}\sum_{0\leqslant p}\left\|\chi_p\big(|\mathbf{y}|\big)\big(1+|\mathbf{y}|\big)^{\frac{3}{2}}\partial_z^l\partial_{\mathbf{y}}^{\alpha}\Phi_j\right\|_{L^2(\mathbb{R}^3_{\mathbf{y}}\times\mathbb{R}^*_z)},$$

$$\chi_0=\mathbf{1}_{[0,1[},\quad 1\leqslant p\Rightarrow\chi_p=\mathbf{1}_{[2^{p-1},2^p[}.$$

The terms with the weight $\frac{1}{\varepsilon}(1+|z|)^{\varepsilon}$ are used to control the contribution of the light Klein–Gordon modes, $\phi_m^{\pm}$, $m<1$, while the terms with the z-derivatives are useful to estimate the heavy modes, $m>1$.

In the other hand, we know that Strichartz has proved in [6] that for suitable initial data, the solutions of the D'Alembertian in the Minkowski space–time $\mathbb{R}^{1+3}$ belong to $L^4(\mathbb{R}^4)$, and the solutions of the massive Klein–Gordon equation belong to $L^q(\mathbb{R}^4)$, $\frac{10}{3}\leqslant q\leqslant 4$. Moreover the massless fields in the Minkowski space–time $\mathbb{R}^{1+4}$ belong to $L^{\frac{10}{3}}(\mathbb{R}^{1+4})$. Therefore, we expect that near the brane, the solutions of the master equation are in $L^{\infty}([-R,R]_z;L^4(\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}))$, and the Kaluza–Klein tower that is more dispersive because of the mass, is in $L^{\frac{10}{3}}(\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times[-R,R]_z)$. We can indeed prove these results by using (3) and the asymptotics $f_m^+(z)\sim-\sqrt{\frac{2}{\pi}}\cos(mz)$, $f_m^-(z)\sim\sqrt{\frac{2}{\pi}}\sin(mz)$, $m\to\infty$.

Theorem 1.3. *For any $R>0$ there exists $C_R>0$ such that for any $\varepsilon\in]0,\frac{1}{2}]$, and for all $\Phi_0\in\mathfrak{K}^0\cap\mathfrak{D}(\mathbf{H}^2)$, $\Phi_1\in\mathfrak{K}^0\cap\mathfrak{D}(\mathbf{H})$, the solution Φ of* (1), (2) *satisfies the following inequalities, provided the norms of the right-hand side are finite*:

$$\left\|\Phi(t,.)\right\|_{L^{\infty}(\mathbb{R}^3_{\mathbf{x}}\times[-R,R]_z)}\leqslant C_R|t|^{-\frac{3}{2}}\sum_{j=0,1}\|\Phi_j\|_{X_{j,\varepsilon}},$$

$$\left\|\Phi(t,\mathbf{x},.)\right\|_{L^{\infty}([-R,R]_z)}\leqslant C_R\big(|\mathbf{x}|+|t|\big)^{-\frac{3}{2}}\sum_{j=0,1}\|\Phi_j\|_{Y_{j,\varepsilon}},$$

$$\|\Phi\|_{L^{\frac{10}{3}}(\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}\times[-R,R]_z)}\leqslant C_R\sum_{j=0,1}\left\|\mathbf{H}^{\frac{1}{2}(\frac{1}{2}-j)}\Phi_j\right\|_{L^2(\mathbb{R}^4)},$$

$$\|\Phi\|_{L^{\infty}([-R,R]_z;L^4(\mathbb{R}_t\times\mathbb{R}^3_{\mathbf{x}}))}\leqslant\frac{C_R}{\sqrt{\varepsilon}}\sum_{j=0,1}\left\|\mathbf{H}^{\frac{1}{2}(1-j)+\varepsilon}\Phi_j\right\|_{L^2(\mathbb{R}^4)}.$$

To develop the scattering theory, we compare the Kaluza–Klein towers with the free waves $\Phi_{\infty}^{\pm}$ with finite energy on the five-dimensional Minkowski space–time:

$$\big(\partial_t^2-\Delta_{\mathbf{x}}-\partial_z^2\big)\Phi_{\infty}^{\pm}=0,\quad(t,\mathbf{x},z)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R},\tag{4}$$

$$\Phi_{\infty}^{\pm}\in C^0\big(\mathbb{R}_t;BL^1\big(\mathbb{R}^4_{\mathbf{x},z}\big)\big),\quad\partial_t\Phi_{\infty}^{\pm}\in C^0\big(\mathbb{R}_t;L^2\big(\mathbb{R}^4_{\mathbf{x},z}\big)\big).\tag{5}$$

The partial Fourier transform with respect to $\mathbf{x}\in\mathbb{R}^3$ allows to reduce the problem to the scattering theory for the $(1+1)$-dimensional Klein–Gordon equations $\partial_t^2u+\mathbf{h}u+|\boldsymbol{\xi}|^2u=0$, $\partial_t^2u-\partial_z^2u+|\boldsymbol{\xi}|^2u=0$, $\boldsymbol{\xi}\in\mathbb{R}^3$, for which we can use the theory by Kato [3].

Theorem 1.4. *For any $\Phi_0\in\mathfrak{K}^1$, $\Phi_1\in\mathfrak{K}^0$, there exist unique free waves $\Phi_{\infty}^{\pm}$ satisfying* (4), (5) *and*

$$\lim_{t\to\pm\infty}\left\|\partial_t\Phi(t)-\partial_t\Phi_{\infty}^{\pm}(t)\right\|_{L^2(\mathbb{R}^4_{\mathbf{x},z})}=0.$$

Moreover, these fields satisfy:

$$\lim_{t\to\pm\infty}\left\|\nabla_{\mathbf{x}}\Phi(t)-\nabla_{\mathbf{x}}\Phi_{\infty}^{\pm}(t)\right\|_{L^2(\mathbb{R}^4_{\mathbf{x},z})}+\left\|\partial_z\Phi(t)+\frac{3}{2}\frac{z}{|z|}\left(\frac{1}{1+|z|}\right)\Phi(t)-\partial_z\Phi_{\infty}^{\pm}(t)\right\|^2_{L^2(\mathbb{R}^4_{\mathbf{x},z})}=0.$$

The wave operators

$$\mathbf{W}^{\pm}:(\Phi_0,\Phi_1)\longmapsto\big(\Phi_{\infty}^{\pm}(0,.),\partial_t\Phi_{\infty}^{\pm}(0,.)\big)$$

are isometries from $\mathfrak{K}^1\times\mathfrak{K}^0$ onto $BL^1(\mathbb{R}^4_{\mathbf{x},z})\times L^2(\mathbb{R}^4_{\mathbf{x},z})$.

The scattering operator $\mathbf{S} := \mathbf{W}^+(\mathbf{W}^-)^{-1}$ is unitary on $BL^1(\mathbb{R}^4_{\mathbf{x},z}) \times L^2(\mathbb{R}^4_{\mathbf{x},z})$ and describes the scattering of the Kaluza–Klein towers by the Minkowski brane. We compute the scattering matrix and we investigate its holomorphic continuation. We recall that the spectral representation $\mathcal{R}$ of the free wave equation in $\mathbb{R}_t \times \mathbb{R}^4_{\mathbf{x},z}$ is defined by

$$\mathcal{R} : (\Phi_0, \Phi_1) \in BL^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \longmapsto \frac{|\sigma|^{\frac{3}{2}}}{\sqrt{2}} \left[i\sigma \mathcal{F}_{\mathbf{x},z}\Phi_0(\sigma\boldsymbol{\omega}) + \mathcal{F}_{\mathbf{x},z}\Phi_1(\sigma\boldsymbol{\omega}) \right] \in L^2(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}}),$$

where $\sigma\boldsymbol{\omega} = (\boldsymbol{\xi}, \zeta) \in \mathbb{R}^3_{\boldsymbol{\xi}} \times \mathbb{R}_\zeta$, and $\mathcal{F}_{\mathbf{x},z}$ is the Fourier transform on $\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}_z$. It will be useful to distinguish the odd and the even part of the scattering operator. Thus we introduce spaces $\boldsymbol{E}_\pm := \{\Phi \in \boldsymbol{E}, \Phi(\mathbf{x}, -z) = \pm\Phi(\mathbf{x}, z)\}$ associated with $\boldsymbol{E} = \mathfrak{K}^1, \mathfrak{K}^0, BL^1, L^2$. It is obvious that $\mathfrak{K}^1_\pm \times \mathfrak{K}^1_\pm$ and $BL^1_\pm \times L^2_\pm$ are let invariant, respectively by the perturbed and the free dynamics. Therefore $\mathbf{S}_\pm$ defined as the restriction of $\mathbf{S}$ to $BL^1_\pm \times L^2_\pm$ is an isometry of this space. We also introduce

$$L^2_\pm(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}}) := \left\{ f \in L^2(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}}),\ f(\sigma, \omega_1, \omega_2, \omega_3, -\omega_4) = \pm f(\sigma, \omega_1, \omega_2, \omega_3, \omega_4) \right\}.$$

Then, $\mathcal{R}$ is an isometry from $BL^1_\pm(\mathbb{R}^4_{\mathbf{x},z}) \times L^2_\pm(\mathbb{R}^4_{\mathbf{x},z})$ onto $L^2_\pm(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}})$. First we establish that the scattering operator $\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-$ is implemented by an explicit scattering amplitude acting as a multiplication operator on $L^2_+(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}}) \oplus L^2_-(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}})$. Finally we show that the singularities of the scattering amplitudes $\sigma \mapsto \mathcal{S}_\pm(\sigma) \in C^0(S^3_{\boldsymbol{\omega}})$, form a lattice of radial half straight lines:

$$\Sigma^{(1)}_\nu := \left\{ z = \alpha z_* \in \widetilde{\mathbb{C}^*},\ 1 \leqslant \alpha,\ H^{(1)}_\nu(z_*) = 0 \right\}, \qquad \Sigma^{(2)}_\nu := \left\{ z = \alpha z_* \in \widetilde{\mathbb{C}^*},\ 1 \leqslant \alpha,\ H^{(2)}_\nu(-z_*) = 0 \right\},$$

with $\nu = 1, 2$.

Theorem 1.5. *There exists $\mathcal{S}_+, \mathcal{S}_- \in C^0(\mathbb{R}^*_\sigma \times S^3_{\boldsymbol{\omega}}; S^1)$ such that for any $f_\pm \in L^2_\pm(\mathbb{R}_\sigma \times S^3_{\boldsymbol{\omega}})$, we have*:

$$\mathcal{R}\mathbf{S}\mathcal{R}^{-1}(f_+ \oplus f_-)(\sigma, \boldsymbol{\omega}) = \mathcal{S}_+(\sigma, \boldsymbol{\omega}) f_+(\sigma, \boldsymbol{\omega}) + \mathcal{S}_-(\sigma, \boldsymbol{\omega}) f_-(\sigma, \boldsymbol{\omega}).$$

Moreover these scattering amplitudes are explicitly known:

$$0 < \sigma \Rightarrow \mathcal{S}_{+[-]}(\sigma, \boldsymbol{\omega}) = +[-] \frac{e^{2i\sigma|\omega_4|}}{i} \frac{H^{(2)}_{1[2]}(\sigma|\omega_4|)}{H^{(1)}_{1[2]}(\sigma|\omega_4|)},$$

$$\sigma < 0 \Rightarrow \mathcal{S}_{+[-]}(\sigma, \boldsymbol{\omega}) = -[+] \frac{e^{2i\sigma|\omega_4|}}{i} \frac{H^{(1)}_{1[2]}(-\sigma|\omega_4|)}{H^{(2)}_{1[2]}(-\sigma|\omega_4|)}.$$

The scattering amplitude $\mathcal{S}_{+[-]}(\sigma, \boldsymbol{\omega})$ considered as a $C^0(S^3_{\boldsymbol{\omega}})$-valued function of $\sigma \in [0, \infty[$ (respectively $\sigma \in]-\infty, 0]$) has an analytic continuation on $\widetilde{\mathbb{C}^*} \setminus \Sigma^{(1)}_{1[2]}$ (respectively $\widetilde{\mathbb{C}^*} \setminus \Sigma^{(2)}_{1[2]}$). For $\sigma_* \in \Sigma^{(j)}_{1[2]}$, there exists $C > 0$ such that $C|\sigma - \sigma_*|^{-1} \leqslant \|\mathcal{S}_{+[-]}(\sigma)\|_{L^\infty(S^3_{\boldsymbol{\omega}})}$ as $\sigma \to \sigma_*$, $\sigma \in \widetilde{\mathbb{C}^*} \setminus \Sigma^{(j)}_{1[2]}$.

All these results of asymptotic behaviours suggest that this brane cosmology model is linearly stable. The non-linear stability of the Minkowski brane is a huge open problem.

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