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Partial Differential Equations

Uniqueness of unbounded solutions of the Lagrangian mean curvature flow equation for graphs

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Abstract

We observe that the comparison result of Barles–Biton–Ley for viscosity solutions of a class of nonlinear parabolic equations can be applied to a geometric fully nonlinear parabolic equation which arises from the graphic solutions for the Lagrangian mean curvature flow. ***To cite this article: J. Chen, C. Pang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Unicité des solutions non bornées du flot lagrangien à courbure moyenne pour les graphes. Nous remarquons que le résultat de comparaison de Barles–Biton–Ley sur les solutions de viscosité d'une classe d'équations non linéaires paraboliques peut être appliqué à une équation géométrique, complètement non linéaire parabolique qui apparaît dans les solutions graphiques pour les flots Lagrangiens à courbure moyenne. ***Pour citer cet article : J. Chen, C. Pang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

We consider the question of uniqueness for the following fully nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \arctan \lambda_j \tag{1}$$

with initial condition $u(x,0) = u_0(x)$, where u is a function from $\mathbb{R}^n$ to $\mathbb{R}$ and λ_j's are the eigenvalues of the Hessian D^2u. This equation arises naturally from geometry. In fact, when u is a regular solution to (1), it is known that the graph $(x, Du(x,t))$ evolves by the mean curvature flow and it is a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic structure, for each t (cf. [5,6]). For a smooth stationary solution to (1), the graph of its gradient is a Lagrangian submanifold with zero mean curvature in $\mathbb{R}^{2n}$. Recently, a smooth longtime entire solution to (1) has been constructed in [2] assuming a certain bound on the Lipschitz norm of Du_0.

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Barles, Biton and Ley have obtained a very useful general comparison result (Theorem 2.1 in [1]) for the viscosity solutions of a class of fully nonlinear parabolic equations, as well as existence result (Theorem 3.1 in [1]). In particular, they showed that (1) admits a unique longtime continuous viscosity solution for any continuous function u_0 in $\mathbb{R}$ when $n = 1$.

In this short note, we observe, via elementary methods, that the hypotheses in the general theorems in [1] are valid for the geometric evolution equation (1) in general dimensions. The result is the following:

Theorem 1.1. *Let u and v be an upper semicontinuous and a lower semicontinuous viscosity subsolution and supersolution to* (1) *in $\mathbb{R}^n \times [0, T)$ respectively. If $u(x, 0) \leqslant v(x, 0)$ for all $x \in \mathbb{R}^n$, then $u \leqslant v$ in $\mathbb{R}^n \times [0, T)$. In particular, for any continuous function u_0 in $\mathbb{R}^n$, there is a unique continuous viscosity solution to* (1) *in $\mathbb{R}^n \times [0, \infty)$.*

2. Hypotheses (H1) and (H2)

We now describe the assumptions in the comparison and existence results in [1].

Let S_n be the linear space of real $n \times n$ symmetric matrices. If $X \in S_n$, there exists an orthogonal matrix P such that $X = P \Lambda P^T$ where Λ is the diagonal matrix with diagonal entries consist of eigenvalues of X. Let Λ^+ be the diagonal matrix obtained by replacing the negative eigenvalues in Λ with 0's. Define $X^+ = P \Lambda^+ P^T$.

Consider a continuous function F from $\mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times S_n$ to $\mathbb{R}$. The following assumptions on F are necessary to apply the results in [1]:

(H1) For any $R > 0$, there exists a function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0^+) = 0$ and

$$F\big(y, t, \eta(x - y), Y\big) - F\big(x, t, \eta(x - y), X\big) \leqslant m_R\big(\eta|x - y|^2 + |x - y|\big)$$

for all $x, y \in \bar{B}(0, R)$ and $t \in [0, T]$, whenever $X, Y \in S_n$ and $\eta > 0$ satisfy

$$-3\eta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant 3\eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

(H2) There exist $0 < \alpha < 1$ and constants $K_1 > 0$ and $K_2 > 0$ such that

$$F(x, t, p, X) - F(x, t, q, Y) \leqslant K_1 |p - q| \big(1 + |x|\big) + K_2 \big(\operatorname{tr}(Y - X)^+\big)^\alpha$$

for every $(x, t, p, q, X, Y) \in \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times S_n \times S_n$.

The operator F is degenerate elliptic if (H2) holds.

Theorem 2.1 *(Barles–Biton–Ley). Let u and v be an upper semicontinuous viscosity subsolution and a lower semicontinuous viscosity supersolution respectively of*

$$\frac{\partial u}{\partial t} + F\big(x, t, Du, D^2 u\big) = 0 \quad \text{in } \mathbb{R}^n \times [0, T),$$
$$u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n.$$

Assume that (H1) *and* (H2) *hold for F. Then*

(1) *If $u(\cdot, 0) \leqslant v(\cdot, 0)$ in $\mathbb{R}^n$, then $u \leqslant v$ in $\mathbb{R}^n \times [0, T)$.*
(2) *If $u_0 \in C(\mathbb{R}^n)$ there is a unique continuous viscosity solution in $\mathbb{R}^n \times [0, \infty)$.*

We now present the proof of Theorem 1.1.

Proof. We define $F : S_n \to \mathbb{R}$ by

$$F(X) = -i \log \frac{\det(I + iX)}{\det(I + X^2)^{\frac{1}{2}}} = -\frac{i}{2} \log \frac{\det(I + iX)}{\det(I - iX)}. \tag{2}$$

That F takes real values follows easily from

$$\overline{F(X)} = \frac{i}{2}\log\frac{\det(I - iX)}{\det(I + iX)} = F(X).$$

Note that $F(D^2u)$, by diagonalizing D^2u at a point, is equal to $\sum \arctan\lambda_j$. Therefore the flow (1) can be written as $u_t + (-F(D^2u)) = 0$.

Since $F(x,t,p,X) = F(X)$ is independent of x, the right-hand side of the inequality for F in (H1) must be zero, namely $m_R = 0$. By multiplying an arbitrary vector $(\xi,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and its transpose to the second matrix inequality in (H1), we see that $X \leqslant Y$. Therefore, in order to establish (H1) it suffices to show:

(H1′) For any $X, Y \in S_n$, if $X \geqslant Y$ then $F(X) \geqslant F(Y)$.

For any $X, Y \in S_n$ and $t \in [0,1]$, define

$$f_{XY}(t) = F\big(tX + (1-t)Y\big).$$

We will show that $f_{XY}(t)$ is nondecreasing in $t \in [0,1]$ and then (H1′) will follow as $f_{XY}(0) = F(Y)$ and $f_{XY}(1) = F(X)$. Set

$$A = I + i\big(tX + (1-t)Y\big)$$

and

$$B = I - i\big(tX + (1-t)Y\big).$$

Then

$$f_{XY}(t) = -\frac{i}{2}(\log\det A - \log\det B).$$

It follows that $AB = BA$ and

$$\big(A^{-1} + B^{-1}\big)\cdot\frac{AB}{2} = \frac{A+B}{2} = I.$$

Note that both A and B are invertible matrices for all $t \in [0,1]$. Hence, by using the formula $\partial_t \ln\det G = \operatorname{tr}(G^{-1}\partial_t G)$ for $G(t) \in GL(n,\mathbb{R})$, we have

$$\begin{aligned} f'_{XY}(t) &= -\frac{i}{2}\operatorname{tr}\big(A^{-1}\cdot\partial_t A - B^{-1}\cdot\partial_t B\big) \\ &= -\frac{i}{2}\operatorname{tr}\big(\big(A^{-1} + B^{-1}\big)\cdot i(X - Y)\big) \\ &= \operatorname{tr}\big(\big(I + \big(tX + (1-t)Y\big)^2\big)^{-1}\cdot(X - Y)\big). \end{aligned} \tag{3}$$

Since $tX + (1-t)Y$ is real symmetric, the matrix

$$C = I + \big(tX + (1-t)Y\big)^2$$

is positive definite, hence so is C^{-1}. There exists a matrix $Q \in GL(n,\mathbb{R})$ such that $C = QQ^T$. By the assumption $X \geqslant Y$, we have

$$\begin{aligned} \operatorname{tr}\big(C^{-1}(X - Y)\big) &= \operatorname{tr}\big(Q\cdot Q^T(X - Y)\big) \\ &= \operatorname{tr}\big(Q^T(X - Y)\cdot Q\big) \\ &\geqslant 0 \end{aligned}$$

since $Q^T(X - Y)Q$ is positive semidefinite. Therefore, we have shown that (H1) is valid for F defined in (2).

As $F(x,t,p,X)$ is independent of p, (H2) reads: there exist constants $K > 0$ and $0 < \alpha < 1$ such that $F(X) - F(Y) \leqslant K(\operatorname{tr}(X - Y)^+)^\alpha$ for all $X, Y \in S_n$.

For any $X, Y \in S_n$, integrating (3) leads to

$$F(X) - F(Y) = \int_0^1 \operatorname{tr}\big(C^{-1}(X - Y)\big)\,\mathrm{d}t. \tag{4}$$

For $X - Y \in S_n$ there exists an orthogonal matrix P such that $X - Y = P\Lambda P^T$ where the diagonal matrix Λ has diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $\lambda_j^+ = \max\{\lambda_j, 0\}$. Since $0 < C^{-1} \leqslant I$, we have $0 < P^T C^{-1} P \leqslant I$. If c_{jj} denote the diagonal entries of $P^T C^{-1} P$ for $j = 1, \ldots, n$, then $c_{jj} = \langle P^T C^{-1} P e_j, e_j\rangle$ where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. It follows that $0 < c_{jj} \leqslant 1$ for $j = 1, \ldots, n$. Then

$$\begin{aligned}\operatorname{tr}\bigl(C^{-1}(X-Y)\bigr) &= \operatorname{tr}\bigl(P^T C^{-1} P \cdot P^T (X-Y) P\bigr)\\ &= \operatorname{tr}\bigl(P^T C^{-1} P \cdot \Lambda\bigr)\\ &= \sum c_{jj}\lambda_j\\ &\leqslant \sum \lambda_j^+\\ &= \operatorname{tr}(X-Y)^+.\end{aligned}$$

Substituting the above inequality into (4) implies: for any $X, Y \in S_n$ we have $F(X) - F(Y) \leqslant \operatorname{tr}(X-Y)^+$.

Because $\arctan x$ is in $(-\pi/2, \pi/2)$, we have $F(X) - F(Y) < n\pi$. For any constant α with $0 < \alpha < 1$, if $\operatorname{tr}(X - Y)^+ \leqslant 1$ then

$$F(X) - F(Y) \leqslant \operatorname{tr}(X-Y)^+ \leqslant n\pi\bigl[\operatorname{tr}(X-Y)^+\bigr]^\alpha$$

and if $\operatorname{tr}(X-Y)^+ > 1$ then

$$F(X) - F(Y) \leqslant n\pi \leqslant n\pi\bigl[\operatorname{tr}(X-Y)^+\bigr]^\alpha.$$

Therefore, (H2) holds for $K_2 = n\pi$ and any constants $K_1 > 0$ and α with $0 < \alpha < 1$.

Now Theorem 1.1 follows immediately from Theorem 2.1. □

We notice that (H1′), for the operator $F(X) = \sum \arctan\lambda_j(X)$, also follows from the basic fact (cf. p. 182 in [4]): Suppose that $X, Y \in S_n$ and the eigenvalues λ_j's of X and μ_j's of Y are in descending order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$. If $X \geqslant Y$, then $\lambda_j \geqslant \mu_j$ for $j = 1, \ldots, n$.

We also mention the uniqueness of viscosity solutions of the Cauchy–Dirichlet problem for (1). Note that the operator $F(X) = \sum \arctan\lambda_j(X)$ satisfies (H1′) which is exactly the fundamental monotonicity condition (0.1) for $-F$ in [3], therefore $-F$ is proper in the sense of [3] (cf. p. 2 in [3]). As (H1) holds, Theorem 8.2 in [3] is valid for (1):

Theorem 2.2. *The continuous viscosity solution to the following Cauchy–Dirichlet problem is unique*:

$$\begin{aligned}&u_t = \sum_{j=1}^n \arctan\lambda_j, \quad \text{in } (0,T)\times\Omega,\\ &u(t,x) = 0, \quad \text{for } 0 \leqslant t < T \text{ and } x \in \partial\Omega,\\ &u(0,x) = \psi(x), \quad \text{for } x \in \overline{\Omega},\end{aligned}$$

where λ_j's are the eigenvalues of D^2u, $\Omega \subset \mathbb{R}^n$ is open and bounded and $T > 0$ and $\psi \in C(\overline{\Omega})$. If u is an upper semicontinuous viscosity subsolution and v is a lower semicontinuous viscosity supersolution of the Cauchy–Dirichlet problem, then $u \leqslant v$ on $[0,T)\times\Omega$.

Note that the initial-boundary conditions for the subsolution and supersolution are: $u(x,t) \leqslant 0 \leqslant v(x,t)$ for $t \in [0,T)$ and $x \in \partial\Omega$ and $u(x,0) \leqslant \psi(x) \leqslant v(v,0)$ for $x \in \overline{\Omega}$.

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