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Complex Analysis

On the duality between $A^{-\infty}(D)$ and $A_D^{-\infty}$ for convex domains

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Abstract

The goal of this Note is to prove that the Laplace transformation of analytic functionals establishes the mutual duality between the spaces $A^{-\infty}(D)$ and $A_D^{-\infty}$ (D being a bounded convex domain in $\mathbb{C}^N$) and that functions from $A_D^{-\infty}$ can be represented in a form of Dirichlet series with frequencies from D. ***To cite this article: A.V. Abanin, L.H. Khoi, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur la dualité entre $A^{-\infty}(D)$ et $A_D^{-\infty}$ pour des domaines convexes. Le but de cette Note est de démontrer que la transformation de Laplace des fonctionnelles analytiques établit une dualité mutuelle entre les espaces $A^{-\infty}(D)$ et $A_D^{-\infty}$ (D étant un domaine convexe borné dans $\mathbb{C}^N$) et que des fonctions de $A_D^{-\infty}$ peuvent être représentées sous la forme de séries de Dirichlet avec fréquence de D. ***Pour citer cet article : A.V. Abanin, L.H. Khoi, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

1.1. Basic notations

$\mathcal{O}(D)$ (D being a domain in $\mathbb{C}^N$) denotes the space of functions holomorphic in D, with the compact-open topology.

$\mathcal{O}(K)$, respectively $C^\infty(K)$ (K being a compact set in $\mathbb{C}^N$), denotes the space of germs of functions holomorphic on K, endowed with the topology of inductive limit, respectively the space of functions infinitely differentiable on K.

If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index from $\mathbb{N}_0^N$ ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), then $|\alpha| = \alpha_1 + \cdots + \alpha_N$. If $z, \zeta \in \mathbb{C}^N$, then $|z| = (z_1\bar{z}_1 + \cdots + z_N\bar{z}_N)^{1/2}$, $\langle z, \zeta\rangle = z_1\zeta_1 + \cdots + z_N\zeta_N$.

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For a set $E \subset \mathbb{C}^N$ $(0 \in E)$ denote $\widetilde{E} := \{w \in \mathbb{C}^N\colon \langle z, w\rangle \neq 1 \text{ for any } z \in E\}$, the conjugate set of E. In the case when E is open, $\widetilde{E}$ is a compact set and plays the role of "the exterior" in the duality of A. Martineau and L. Aizenberg [1,7].

1.2. *The function algebra $A^{-\infty}(D)$*

Let D be a bounded domain in $\mathbb{C}^N$. Put $\mathrm{d}(\lambda) = \inf_{\zeta\in\partial D}|\lambda - \zeta|$, $\lambda \in D$, the minimum Euclidean distance between λ and the boundary ∂D of D. The space $A^{-\infty}(D)$ is defined as follows:

$$A^{-\infty}(D) = \Big\{ f \in \mathcal{O}(D)\colon \exists n, C > 0,\ \sup_{\lambda\in D} \big|f(\lambda)\big|\big[\mathrm{d}(\lambda)\big]^n \leqslant C \Big\}.$$

Notice that the condition in the definition of $A^{-\infty}(D)$ is the familiar polynomial growth condition $\sup_{\lambda\in D}(1 - |\lambda|)^n|f(\lambda)| \leqslant C$ if the domain D is the open unit ball.

The space $A^{-\infty}(D)$ can be thought of as the union of the Banach spaces

$$A^{-n}(D) = \Big\{ f \in \mathcal{O}(D)\colon \|f\|_n = \sup_{\lambda\in D} \big|f(\lambda)\big|\big[\mathrm{d}(\lambda)\big]^n < +\infty \Big\}.$$

We can endow $A^{-\infty}(D)$ with a natural topology of inductive limit of spaces $A^{-n}(D)$.

1.3. *The function space $A_D^{-\infty}$*

Let D be convex. Without loss of generality, we can assume that $0 \in D$. Define a space

$$A_D^{-\infty} = \Big\{ f \in \mathcal{O}\big(\mathbb{C}^N\big)\colon |f|_n = \sup_{z\in\mathbb{C}^N} \frac{|f(z)|(1+|z|)^n}{\exp H_D(z)} < \infty,\ \forall n \in \mathbb{N} \Big\},$$

where H_D is the supporting function of D, endowed with the topology given by the system of norms $(|\cdot|_n)_{n=1}^{\infty}$.

1.4. *The goal of the Note*

In this Note we establish, via the Laplace transformation, the mutual duality between $A^{-\infty}(D)$ and $A_D^{-\infty}$. As an application of the obtained result, a representation of functions from $A_D^{-\infty}$ in a form of Dirichlet series with frequencies from D is also studied.

It should be noted that the duality problem for the space $A^{-\infty}(D)$ has been studied by several authors, and by different methods. In particular, S. Bell, E. Straube, D. Barrett, C. Kiselman (see, e.g., [2,9] and references therein), established the duality between $A^{-\infty}(D)$ and the space $A^{\infty}(\overline{D})$ of holomorphic functions in D that are in $C^{\infty}(\overline{D})$, and therefore, their results are quite different from ours. Also, the representation is never treated in above-mentioned papers.

2. Statement of the main results

Throughout the rest of this Note, let D be either a bounded convex domain with C^2 boundary in $\mathbb{C}^N$ when $N > 1$, or an arbitrary bounded convex domain in $\mathbb{C}$.

2.1. *The duality problem*

The Laplace transformation of an analytic functional φ on the space $A_D^{-\infty}$, or respectively, on $A^{-\infty}(D)$, is defined as $\mathcal{F}(\varphi)(\lambda) := \varphi_z(e^{\langle z,\lambda\rangle})$, $\varphi \in (A_D^{-\infty})'$, $\lambda \in D$, or respectively, $\mathcal{F}(\varphi)(z) := \varphi_\lambda(e^{\langle z,\lambda\rangle})$, $\varphi \in (A^{-\infty}(D))'$, $z \in \mathbb{C}^N$.

Theorem 2.1. *The Laplace transformation establishes a topological isomorphism between the following spaces*:

(a) *The strong dual $(A_D^{-\infty})'_b$ of $A_D^{-\infty}$ and the space $A^{-\infty}(D)$.*
(b) *The strong dual $(A^{-\infty}(D))'_b$ of $A^{-\infty}(D)$ and the space $A_D^{-\infty}$.*

Note that for $N = 1$ part (b) was also obtained by S. Melikhov in [8].

2.2. The representation problem

In a general setting, a sequence (x_k) of elements of a locally convex space H is said to be an *absolutely representing system* in H if any element x from H can be represented in a form of the series $x = \sum c_k x_k$, which converges absolutely in the topology of H. This theory finds, in particular, important applications to functional equations, say representation of solutions in series of simpler functions, like exponential functions, or rational functions. We refer the reader to [6] and references therein, for more detailed information.

Theorem 2.2. *There is an explicit construction of $(\lambda_k)_{k=1}^{\infty} \subset D$, such that the system $(e^{\langle \lambda_k, z\rangle})_{k=1}^{\infty}$ is absolutely representing in the space $A_D^{-\infty}$, that is, any function $f \in A_D^{-\infty}$ can be represented in a form of Dirichlet series*

$$f(z) = \sum_{k=1}^{\infty} c_k e^{\langle \lambda_k, z\rangle}, \quad \forall z \in \mathbb{C}^N,$$

that converges absolutely in the space $A_D^{-\infty}$.

3. Sketch of proofs

3.1. For Theorem 2.1

The most difficult part is to show the surjectivity of $\mathcal{F}$.

First for $N > 1$, denote $\rho(z) = \begin{cases} -\mathrm{d}(z), z \in D \\ \mathrm{d}(z), z \notin D \end{cases}$. Since D has C^2 boundary, $\rho(z) \in C^2$ in some neighborhood of ∂D. For $\delta > 0$ sufficiently small, $\rho \in C^2(\overline{D} \setminus D_\delta)$, where $D_\delta = \{z \in D\colon\ \mathrm{d}(z) > \delta\}$.

Put

$$\nabla_z \rho = \left(\frac{\partial \rho}{\partial z_1}, \ldots, \frac{\partial \rho}{\partial z_N}\right); \qquad R_j(z) = \det \begin{pmatrix} \frac{\partial \rho}{\partial z_1} & \cdots & \frac{\partial \rho}{\partial z_N} \\ \frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_1} & \cdots & \frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_N} \\ \cdots & [j] & \cdots \\ \frac{\partial^2 \rho}{\partial \bar{z}_N \partial z_1} & \cdots & \frac{\partial^2 \rho}{\partial \bar{z}_N \partial z_N} \end{pmatrix};$$

$$\bar{\omega}(z, \nabla_z \rho) = \langle z, \nabla_z \rho\rangle^{-N} \sum_{j=1}^{N} R_j(z)\, \mathrm{d}\bar{z}_1 \wedge \cdots [j] \cdots \wedge \mathrm{d}\bar{z}_N \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_N;$$

$$u(z) = \langle z, \nabla_z \rho\rangle^{-1} \nabla_z \rho = \big(u_1(z), \ldots, u_N(z)\big);$$

$$\mathcal{R}_j(z) = \langle z, \nabla_z \rho\rangle^{-N} \sum_{k=1}^{N} \frac{\partial \bar{u}_j}{\partial \bar{z}_k}(z)(-1)^{k-1} R_k(z), \quad j = 1, \ldots, N.$$

For $f \in A_D^{-\infty}$ construct

$$F(u) := \frac{\xi^{N-1}}{(N-1)!} \int_0^{\infty} f(t\xi u) t^{N-1} e^{-t\xi}\, \mathrm{d}t,$$

where $u = \gamma w$ ($0 \leqslant \gamma \leqslant 1$, w is an arbitrary point of $\partial \widetilde{D}$), $\xi \in \mathbb{C}$ with $|\xi| = 1$ and $\operatorname{Re} \xi > 0$ is chosen so that $H_D(\xi w) = \operatorname{Re} \xi$. By [7, Lemma 21], F is holomorphic in $\int \widetilde{D}$. In our case it can be proved that F is infinitely differentiable on $\widetilde{D}$ as a function of $2N$ real variables and

$$\big|F^{(\alpha)}(u)\big| \leqslant A|f|_n, \quad \forall u \in \widetilde{D},\ \forall n \geqslant |\alpha| + N + 1,$$

where A is a constant depending only on $|\alpha|$, n, N and D.

(a) Let $g \in A^{-\infty}(D)$. For each $\gamma \in (0, 1)$ define

$$\langle g, f\rangle_\gamma := \frac{(N-1)!}{(2\pi i)^N} \int_{\partial D} g(\gamma z) F\big(u(z)\big) \bar{\omega}(z, \nabla_z \rho), \quad f \in A_D^{-\infty}.$$

According to Whitney's extension theorem (see, e.g., [4, Theorem 2.3.6]), for each $m \in \mathbb{N}$ there exists a linear continuous extension operator $\mathcal{L}: C^m(\widetilde{D}) \to C_0^m(\widetilde{D}_\delta)$. By Green–Stokes formula, we have

$$\langle g, f\rangle_\gamma = \frac{(N-1)!}{(2\pi i)^N} \int\limits_{D\setminus D_\delta} g(\gamma z) \sum_{j=1}^{N} \frac{\partial(\mathcal{L}F)}{\partial \bar{u}_j}\big(u(z)\big)\mathcal{R}_j(z)\,\mathrm{d}\bar{z}\wedge \mathrm{d}z. \tag{1}$$

Taking into account that $\bar{\partial}F = 0$ on $\widetilde{D}$ and using Taylor formula, continuity of $\mathcal{L}$ and the above-mentioned estimate of $|F^{(\alpha)}|$, we find that

$$\big|\langle g, f\rangle_\gamma\big| \leqslant C\|g\|_m |f|_n, \quad \forall n \geqslant m+N+1,\ \forall f \in A_D^{-\infty},$$

where C depends only on m, n, N and D. Hence, $\langle g, \cdot\rangle_\gamma \in (A_D^{-\infty})'$. From (1) it follows that there exists the limit

$$\lim_{\gamma\uparrow 1}\langle g, f\rangle_\gamma = \frac{(N-1)!}{(2\pi i)^N} \int\limits_{D\setminus D_\delta} g(z) \sum_{j=1}^{N} \frac{\partial(\mathcal{L}F)}{\partial \bar{u}_j}\big(u(z)\big)\mathcal{R}_j(z)\,\mathrm{d}\bar{z}\wedge \mathrm{d}z.$$

By Banach–Steinhaus theorem, $\langle g, \cdot\rangle := \lim_{\gamma\uparrow 1}\langle g, \cdot\rangle_\gamma \in (A_D^{-\infty})'$. Applying Leray's integral formula, we obtain that $\langle g, e^{\langle \cdot, \lambda\rangle}\rangle = g(\lambda)$, $\forall \lambda \in D$.

(b) Now let $f \in A_D^{-\infty}$. By Whitney's extension theorem for C^∞-functions ([10, Theorem I]), F can be extended to a function $\widetilde{F} \in C_0^\infty(\widetilde{D}_\delta)$ so that $(\widetilde{F}|_{\widetilde{D}})^{(\alpha)} = F^{(\alpha)}$, $\forall\alpha$. Using $\bar{\partial}F = 0$ on $\widetilde{D}$ and Taylor formula again, we find that

$$\langle g, f\rangle := \frac{(N-1)!}{(2\pi i)^N} \int\limits_{D\setminus D_\delta} g(z) \sum_{j=1}^{N} \frac{\partial \widetilde{F}}{\partial \bar{u}_j}\big(u(z)\big)\mathcal{R}_j(z)\,\mathrm{d}\bar{z}\wedge \mathrm{d}z, \quad g \in A^{-\infty}(D),$$

is a continuous linear functional on $A^{-\infty}(D)$. It remains to apply Martineau's projective formula to obtain $\langle e^{\langle z, \cdot\rangle}, f\rangle = f(z)$, $\forall z \in \mathbb{C}^N$.

Next, for $N = 1$ we can see that $\langle g, f\rangle := \lim_{\gamma\uparrow 1} \frac{1}{2\pi i}\int_{\partial D} g(\gamma z) F(\frac{1}{z})\frac{\mathrm{d}z}{z}$ and $\langle g, f\rangle := \frac{1}{2\pi i}\int_D g(z)\frac{\partial \widetilde{F}}{\partial \bar{z}}\,\mathrm{d}\bar{z}\wedge \mathrm{d}z$, where $\widetilde{F}$ is a C^∞-extension of F in $\mathbb{R}^2$, work well.

3.2. *For Theorem 2.2*

Construct a sequence $(\lambda_k)_{k=1}^\infty \subset D$ by a method in [3, Theorem 4.5] (see also [5, Theorem 3.1]), which forms the so-called weakly sufficient set in $A^{-\infty}(D)$. The result follows from Theorem 2.1(a) and [6, Corollary of Theorem F].

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