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A new variational approach to the stability of gravitational systems

Mohammed Lemou [a], Florian Méhats [b], Pierre Raphaël [c]

[a] CNRS and IRMAR, université de Rennes 1, 35042 Rennes, France
[b] IRMAR, université de Rennes 1, 35042 Rennes, France
[c] IMT, université Paul-Sabatier, 31062 Toulouse cedex 9, France

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Abstract

We consider the three-dimensional gravitational Vlasov–Poisson system which describes the mechanical state of a stellar system subject to its own gravity. A well-known conjecture in astrophysics is that the steady state solutions which are nonincreasing functions of their microscopic energy are nonlinearly stable by the flow. This was proved at the linear level by Antonov in 1961. Since then, standard variational techniques based on concentration compactness methods as introduced by P.-L. Lions in 1984 have led to the nonlinear stability of subclasses of stationary solutions of ground state type. In this Note, we propose a new variational approach based on the minimization of the Hamiltonian under equimeasurable constraints, which are conserved by the nonlinear transport flow, and recognize any steady state solution which is a nonincreasing function of its microscopic energy as a local minimizer. The outcome is the proof of its nonlinear stability under radially symmetric perturbations. ***To cite this article: M. Lemou et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une nouvelle approche variationnelle pour la stabilité des systèmes gravitationnels. Nous considérons le système de Vlasov–Poisson gravitationnel qui décrit l'état d'un système stellaire soumis à la seule force de gravitation. Une conjecture classique en astrophysique est que les états stationnaires du système qui sont des fonctions strictement décroissantes de leur énergie microscopique sont non-linéairement stables par le flot. Ceci fut démontré dès 1961 par Antonov au niveau linéaire. Depuis, l'application des techniques variationelles de concentration compacité telles qu'introduites par P.-L. Lions in 1984 a permis de prouver la stabilité orbitale de certaines sous-classes de profils de type état fondamental. Dans cette Note, nous proposons une nouvelle approche variationnelle basée sur la minimisation du Hamiltonien sous des contraintes d'équimesurabilité, qui sont préservées par le transport non linéaire. Nous démontrons que tout état stationnaire qui est une fonction décroissante de son énergie microscopique est un minimiseur local, ce qui implique sa stabilité non-linéaire contre des perturbations à symétrie sphérique. ***Pour citer cet article : M. Lemou et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

E-mail addresses: mohammed.lemou@univ-rennes1.fr (M. Lemou), florian.mehats@univ-rennes1.fr (F. Méhats), pierre.raphael@math.univ-toulouse.fr (P. Raphaël).

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Nous considérons le système de Vlasov–Poisson gravitationnel

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla\phi_f \cdot \nabla_v f = 0, & (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f(t=0,x,v) = f_0(x,v) \geqslant 0, & \rho_f(x) = \int_{\mathbb{R}^3} f(x,v)\,\mathrm{d}v, \quad \phi_f(x) = -\frac{1}{4\pi|x|} * \rho_f. \end{cases} \tag{1}$$

Cette équation de transport non-linéaire fournit une description cinétique de l'état d'un système stellaire soumis à la seule force de gravitation.

Une question importante en astrophysique est celle de la stabilité des solutions stationnaires. Le théorème de Jeans [2], classifie les états stationnaires à symétrie sphérique comme étant de la forme :

$$Q(x,v) = F(e,\ell), \quad e = \frac{|v|^2}{2} + \phi_Q\big(|x|\big), \quad \ell = |x \times v|^2. \tag{2}$$

Nous supposerons en outre que Q est un profil strictement décroissant au sens suivant :

(H): Q est continu et à support compact avec $\frac{\partial F}{\partial e} < 0$ sur son support.

Une conjecture classique en astrophysique [3] est que, sous (H), Q est stable par le flot de (1) sous perturbations radiales.

Une démonstration remarquable de ce résultat *au niveau linéaire* est donnée par Antonov [1], voir aussi Lynden-Bell [12]. Depuis, une application standard des techniques variationelles de concentration-compacité, [10], a permis d'obtenir la stabilité de sous-classes de solutions stationaires de type état fondamental, [5,7,9,14]. La faiblesse de ces techniques est de considérer la minimisation du Hamiltonien contre un nombre fini de contraintes alors qu'une infinité de contraintes est à disposition grâce au transport non-linéaire.

Nous proposons dans cette Note une approche variationnelle généralisée permettant de traiter la stabilité de tous les états stationnaires Q de la forme (2) vérifiant (H). Soit une fonction de distribution $f(x,v)$ dans l'espace d'énergie

$$\mathcal{E}_{rad} = \big\{ f \in L^1 \cap L^\infty,\ |v|^2 f \in L^1,\ f \geqslant 0,\ f \text{ à symétrie sphérique} \big\},$$

et sa fonction de distribution à moment cinétique donné :

$$\mu_f(s,\ell) = \nu_l\big\{(x,v) \in \mathbb{R}^6\colon\ f(x,v) > s,\ |x \times v|^2 = \ell\big\},$$

où la mesure ν_ℓ considérée est une projection explicite de la mesure de Lebesgue sur la variété $|x \times v|^2 = \ell$, voir (8). Soit l'espace d'états :

$$Eq(f) = \{g \in \mathcal{E}_{rad} \text{ avec } \mu_g = \mu_f\}.$$

Pour des données régulières *à symétrie sphérique*, le transport non-linéaire (1) assure le transport des fonctions de distribution à moment cinétique donné :

$$\forall t \geqslant 0, \quad \mu_{f(t)} = \mu_{f_0} \quad \text{i.e.}; \quad f(t) \in Eq(f_0), \tag{3}$$

ainsi que la conservation globale du Hamiltonien :

$$\forall t \geqslant 0, \quad \mathcal{H}\big(f(t)\big) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f(t)\,\mathrm{d}x\,\mathrm{d}v - \frac{1}{2}\int_{\mathbb{R}^3} \big|\nabla\phi_{f(t)}\big|^2 = \mathcal{H}(f_0). \tag{4}$$

Notre premier résultat est la caractérisation variationnelle locale de Q :

Théorème 0.1 *(Caractérisation variationnelle de Q). Soit Q vérifiant* (H)*, alors Q est un minimiseur local strict du Hamiltonien sous la contrainte $f \in Eq(Q)$.*

Une étude plus poussée des suites minimisantes et les lois de conservation (3), (4) nous permettent d'en déduire la stabilité non-linéaire de Q :

Théorème 0.2 *(Stabilité de Q sous perturbations à symétrie sphérique). Soit Q vérifiant* (H)*, alors Q est non-linéairement stable par le flot de* (1) *pour des perturbations régulières à symétrie sphérique.*

La conjecture de stabilité par (1) des états stationnaires décroissants est donc démontrée pour des perturbations à symétrie sphérique. Un problème majeur à la suite de ce travail est le cas des perturbations non radiales. Cette note est une version abrégée de [8].

1. Introduction and main results

We study the gravitational Vlasov–Poisson system (1). This nonlinear transport equation describes the mechanical state of a stellar system subject to its own gravity. An important issue both from the mathematical and physical point of view is the stability of steady states solutions. Jeans' theorem, [2], classifies all spherically symmetric stationary solutions to (1) as of the form:

$$Q(x,v)=F(e,\ell),\quad e=\frac{|v|^2}{2}+\phi_Q\big(|x|\big),\quad \ell=|x\times v|^2. \tag{5}$$

We shall, moreover, assume that Q is a nonincreasing profile in the following sense:

(H): Q is continuous and compactly supported with $\frac{\partial F}{\partial e}<0$ on its support.

A classical conjecture in astrophysics is that (H) is enough to ensure the stability of Q by the flow (1) under radial perturbations, see [3].

Remarkably enough, a proof of this conjecture *at the linear level* goes back to Antonov in 1961, [1], see also Lynden-Bell [12]. For the full nonlinear problem, variational techniques using standard concentration compactness techniques as introduced by Lions in the 1980s, [10], yield the orbital stability of ground state type stationary solutions, see in particular [5,7,9,14]. The weakness of these techniques however is that they rely on the minimization of the conserved Hamiltonian of (1) under *finitely many* constraints while in fact a continuum of constraints is at hand thanks to the nonlinear transport. In [6,4], a different strategy is designed to derive the stability of the so called King model, which was not reached by classical variational techniques. This strategy relies on a fine study of the linearized operator in the continuation of the works [11]. This interesting approach is however so far limited to the specific King model.

Our aim in this Note is to implement a generalized variational approach taking full account of the nonlinear transport and allowing one to prove the nonlinear stability of *all* steady solutions Q satisfying (H) against radially symmetric perturbations. Given a distribution function $f\geqslant 0$ in the energy space

$$\mathcal{E}_{rad}=\big\{f\in L^1\cap L^\infty,\ |v|^2f\in L^1,\ f\geqslant 0,\ f \text{ spherically symmetric}\big\},$$

let us recall the invariants of the flow (1) for smooth initial data. There first holds the global constraint of the conservation of the Hamiltonian (4). Next, in the radial setting, the quantity $\ell=|x\times v|^2$ is conserved by the characteristic flow associated to (1) and hence: $\forall G(s,\ell)\in\mathcal{C}^1(\mathbb{R}_+\times\mathbb{R}_+)$ with $G(0,\ell)=0$,

$$\int_{\mathbb{R}^6} G\big(f(t,x,v),|x\times v|^2\big)\,\mathrm{d}x\,\mathrm{d}v=\int_{\mathbb{R}^6} G\big(f(0,x,v),|x\times v|^2\big)\,\mathrm{d}x\,\mathrm{d}v. \tag{6}$$

Now as a simple change of variables formula reveals, there holds:

$$\int_{\mathbb{R}^6} G\big(f(x,v),|x\times v|^2\big)\,\mathrm{d}x\,\mathrm{d}v=\int_{\ell,s>0}\partial_1 G(s,\ell)\mu_f(s,\ell)\,\mathrm{d}s\,\mathrm{d}\ell \tag{7}$$

where $\mu_f(s,\ell)$ is the distribution function at fixed ℓ:

$$\mu_f(s,\ell)=4\pi^2\int_{r>0}\int_{u\in\mathbb{R}}\mathbb{1}_{f(r,u,\ell)>s}\mathbb{1}_{r^2u^2>\ell}\frac{r|u|}{\sqrt{r^2u^2-\ell}}\,\mathrm{d}r\,\mathrm{d}u. \tag{8}$$

Hence if we define the set of states:

$$Eq(f)=\big\{g\in L^1\cap L^\infty,\ |v|^2g\in L^1,\ g\geqslant 0,\ g \text{ spherically symmetric},\ \mu_g=\mu_f\big\}, \tag{9}$$

then the conservation laws (6) are equivalent to:

$$\forall t \geqslant 0, \quad f(t) \in Eq(f_0).$$

Our first main result is a variational characterization of the steady states Q satisfying (H) as strict local minimizers of the Hamiltonian under the equimeasurability constraint $f \in Eq(Q)$:

Theorem 1.1 *(Variational characterization of nonincreasing steady states). Let Q be a spherically symmetric steady state to* (1) *satisfying* (H). *Then there exists a constant $C_0 > 0$ such that the following holds. For all $R > 0$, there exists $\delta_0(R) > 0$ such that, for all $f \in \mathcal{E}_{rad} \cap Eq(Q)$ satisfying $|f - Q|_{\mathcal{E}} \leqslant R$, $|\nabla\phi_f - \nabla\phi_Q|_{L^2} \leqslant \delta_0(R)$, we have*:

$$\mathcal{H}(f) - \mathcal{H}(Q) \geqslant C_0 |\nabla\phi_f - \nabla\phi_Q|_{L^2}^2 \quad \textit{and} \quad \mathcal{H}(f) = \mathcal{H}(Q) \quad \textit{iff} \quad f = Q. \tag{10}$$

This theorem should be thought of as the generalization of well-known results for the simpler transport setting of the two-dimensional incompressible Euler system. These steady states with suitable monotonicity properties of the vorticity profile are minimizers of well chosen conserved quantities and this implies their nonlinear stability, see [13]. The situation for Vlasov–Poisson is however much more intricate due in particular to the nontrivial structure of the steady states solutions (5).

Our proof relies on a new *monotonicity property* of the Hamiltonian under a generalized Schwarz symmetrization which is not the standard radial rearrangement but *a rearrangement with respect to a given microscopic energy* $\frac{|v|^2}{2} + \phi(x)$ at fixed angular momentum $|x \times v|^2$. There are two unexpected outcomes of this strategy. The first one is that it reduces the understanding of the minimization problem (10) to a *new unconstrained minimization problem on the Poisson field ϕ_f only*. Now Q is a critical point of the corresponding functional from (10), and the second derivative of the functional at Q is related to a *Hartree–Fock exchange operator*, whose structure is similar to that introduced in [12] and which is coercive from Antonov's stability criterion. Note that this shades a new light on this operator which already appeared in [4], but as a singular limit of the linearized transport flow close to Q. Second, by reducing the problem to a problem on the Poisson field only, we are able thanks to the radial setting to extract *compactness* from any minimizing sequence, and this yields the full stability of Q through the flow (1) under spherically symmetric perturbations:

Theorem 1.2 *(Nonlinear stability of Q under the nonlinear flow (1)). Let Q be a spherically symmetric steady state to* (1) *satisfying* (H). *Then for all $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds true. Let $f_0 \in \mathcal{E}_{rad}$ smooth enough with*:

$$|f_0 - Q|_{L^1} < \eta, \quad |f_0|_{L^\infty} < |Q|_{L^\infty} + 1, \quad \mathcal{H}(f_0) < \mathcal{H}(Q) + \eta, \tag{11}$$

then the corresponding solution $f(t)$ to (1) *satisfies*:

$$\forall t > 0, \quad \big|f(t) - Q\big|_{L^1} < \varepsilon, \quad \big|f(t)\big|_{L^\infty} < |Q|_{L^\infty} + 1, \quad \big||v|^2\big(f(t) - Q\big)\big|_{L^1} < \varepsilon. \tag{12}$$

Hence the conjecture of stability of nonincreasing radially symmetric steady states is proved for radial perturbations. The main open problem after this work is certainly the general setting of nonradial perturbations. We also expect our strategy to be quite robust and to apply to a large class of nonlinear transport problems. This note is an abridged version of [8].

2. Overview of the proof of Theorem 1.1

Step 1. Symmetric rearrangement with respect to a given microscopic energy.

Let $f \in \mathcal{E}_{rad}$ and ϕ be a given Poisson field, we look for a distribution function g which is a function of $\ell = |x \times v|^2$ and a nonincreasing function of the microscopic energy $e = \frac{|v|^2}{2} + \phi(x)$, and which is equimeasurable to f i.e. $g \in Eq(f)$. Eqs. (7) and (8) naturally lead to the following definition:

Proposition 2.1 *(Symmetric rearrangement with respect to a given microscopic energy). Let $f \in \mathcal{E}_{rad}$ and f^* denote its generalized nonincreasing Schwarz rearrangement at fixed ℓ:*

$$\text{for a.e. } \ell > 0, \quad \forall t \geqslant 0, \quad f^*(t,\ell) = \begin{cases} \sup\{s \geqslant 0\colon\ \mu_f(s,\ell) > t\} & \text{for } t < \mu_f(0,\ell),\\ 0 & \text{for } t \geqslant \mu_f(0,\ell).\end{cases}$$

Let also ϕ be a given Poisson field – in a suitable functional space. We define

$$a_\phi(e,\ell) = 8\pi^2\sqrt{2}\int_0^{+\infty}\left(e - \phi(r) - \frac{\ell}{2r^2}\right)_+^{1/2} \mathrm{d}r$$

where $a_+ = \max\{a, 0\}$, and consider the new distribution function:

$$f^{*\phi}(x,v) = f^*\left(a_\phi\left(\frac{|v|^2}{2} + \phi(x), |x \times v|^2\right), |x \times v|^2\right)\mathbf{1}_{\frac{|v|^2}{2}+\phi(x)<0}. \tag{13}$$

Then

$$f^{*\phi} \in Eq(f).$$

Step 2. Monotonicity of the Hamiltonian under the $f^{*\phi_f}$ rearrangement.

A simple observation is that all ground states Q *satisfying* (H) are fixed points of the rearrangement (13):

$$Q^{*\phi_Q} = Q. \tag{14}$$

A fundamental issue is now that the Hamiltonian $\mathcal{H}$ given by (4) is strictly monotone under (13):

Proposition 2.2 *(Strict monotonicity of the Hamiltonian under the $f^{*\phi_f}$ rearrangement). Let $f \in \mathcal{E}_{rad}$, then*:

$$\mathcal{H}(f) \geqslant \mathcal{H}\big(f^{*\phi_f}\big) \quad \text{with} \quad \mathcal{H}(f) = \mathcal{H}\big(f^{*\phi_f}\big) \quad \text{iff} \quad f = f^{*\phi_f}. \tag{15}$$

Step 3. Reduction to a problem on ϕ_f and coercivity of the Antonov functional.

Let now $f \in Eq(Q)$ so that $f^* = Q^*$, a careful analysis of the monotonicity (15) yields the lower control:

$$\mathcal{H}(f) - \mathcal{H}(Q) \geqslant \mathcal{H}\big(Q^{*\phi_f}\big) + \frac{1}{2}|\nabla\phi_f - \nabla\phi_{Q^{*\phi_f}}|_{L^2}^2 = \mathcal{J}(\phi_f), \tag{16}$$

the key observation being that the functional $\mathcal{J}$ only depends on ϕ_f. We now claim that any Q satisfying (H) is such that ϕ_Q is a *strict* local minimizer of $\mathcal{J}$:

Proposition 2.3 *(ϕ_Q is a strict local minimizer of $\mathcal{J}$). There exist a constant $C_0 > 0$ such that the following holds. For all $R > 0$, there exists $\delta_0(R) > 0$ such that, for all $f \in \mathcal{E}_{rad}$ satisfying $|f - Q|_{\mathcal{E}} \leqslant R$, $|\nabla\phi_f - \nabla\phi_Q|_{L^2} \leqslant \delta_0(R)$, we have*

$$\mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q) \geqslant C_0|\nabla\phi_f - \nabla\phi_Q|_{L^2}^2. \tag{17}$$

This result relies on a Taylor expansion of $\mathcal{J}$ near ϕ_Q. The fact that ϕ_Q is a critical point of $\mathcal{J}$ follows from the steady state equation (14). Now the Hessian of $\mathcal{J}$ at ϕ_Q is intimately connected to the Lynden-Bell Hartree–Fock exchange operator introduced in [12] and the remarkable Antonov coercivity property derived in [1]. The strict positive definitiveness of the Hessian follows and implies (17). Theorem 1.1 is now a straightforward consequence of (15), (16) and (17).

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