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On some relativistic-covariant stochastic processes in Lorentzian space-times

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He querido proceder por definición, no por suposición [...].
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Abstract

A Lorentz-covariant relativistic Brownian motion has been defined by Dudley in the framework of special relativity, and extended to general relativity by Franchi and Le Jan. It is a random timelike curve of class C^1, the world-line of a particle with an intrinsic (i.e., relativistically covariant) law of motion. Building on the Franchi–Le Jan process, we propose a possible definition for random spacelike curves enjoying relativistic covariance; they are more regular (at least C^2) than the timelike ones. ***To cite this article: M. Émery, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Sur certains processus stochastiques à covariance relativiste dans les espaces-temps lorentziens. Défini en relativité restreinte par Dudley, le mouvement brownien relativiste à covariance lorentzienne a été étendu à la relativité générale par Franchi et Le Jan. C'est une courbe aléatoire de classe C^1, de genre temps, et dont la dynamique jouit de la covariance relativiste. En utilisant le processus de Franchi et Le Jan, nous définissons des courbes aléatoires de genre espace et à dynamique covariante. Ces courbes sont plus régulières (C^2 au moins) que celles de Franchi et Le Jan. ***Pour citer cet article : M. Émery, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The mathematical Brownian motion B, as introduced by A. Einstein [2] (independently of earlier work by L. Bachelier), is a non-relativistic model of the physical Brownian motion. It does not have Galilean covariance, for the frames where the ambient fluid is at rest play a special role; adding a constant drift to B changes its dynamics. Similarly, for the same reason, relativistic models of physical Brownian motion [5] cannot be Lorentz covariant. But, if B is a classical Brownian motion in $\mathbb{R}^d$, the process $Y_t = \int_0^t B_s \,\mathrm{d}s$, with velocity B and with C^1 paths, obeys a Galilean-

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covariant dynamics. In 1966, R.M. Dudley [1] considered a relativistic analogue to Y, a mathematical stochastic process, whose paths are C^1 timelike curves (i.e., world-lines) in the Minkowski space-time, and whose dynamics is Lorentz covariant.

Let $\mathbb{E}$ be a d-dimensional Euclidean vector space, and $\mathbb{M} = \mathbb{R} \times \mathbb{E}$ the associated $(d+1)$-dimensional Minkowski space-time, with generic element $x = (x^0, x^{\to})$, and with the Lorentzian scalar product $q(x, y) = x^0 y^0 - x^{\to} \cdot y^{\to}$. The unitary elements of $\mathbb{M}$ form a d-dimensional manifold with three connected components: $\mathbb{H}^+ = \{x \in \mathbb{M}\colon\ q(x,x) = 1 \text{ and } x^0 > 0\}$, $\mathbb{H}^- = \{x \in \mathbb{M}\colon\ q(x,x) = 1 \text{ and } x^0 < 0\}$, and $\mathbb{H}^{\mathrm{sp}} = \{x \in \mathbb{M}\colon\ q(x,x) = -1\}$. (There is one exception, though: when $d = 1$, $\mathbb{H}^{\mathrm{sp}}$ is no longer a connected hyperboloid of one sheet, but a hyperbola, with two connected components.) The submanifolds $\mathbb{H}^+$ and $\mathbb{H}^-$ of $\mathbb{M}$ are spacelike, so the geometry they inherit from $\mathbb{M}$ is Riemannian (with a negative definite metric); if $d > 1$ they have constant curvature -1. The submanifold $\mathbb{H}^{\mathrm{sp}}$ is timelike, its induced structure is Lorentzian, with constant curvature -1 (when $d = 1$, it becomes Riemannian).

Dudley's Brownian motion is the random timelike curve x_t in $\mathbb{M}$ defined as follows: it is parametrized by its proper time t; and its direction $\dot{x}_t$ at time t, which is a "four-vector" in $\mathbb{H}^+$, is a Riemannian Brownian motion in the hyperbolic space $\mathbb{H}^+$. It is convenient to introduce a scale parameter σ, because the speed of light has been set to 1, thus providing a relation between lengths and durations; so the diffusion coefficient of the Brownian motion cannot be arbitrarily set to 1 too (as is usually done in the classical setting), for this would impose another such relation and kill the scaling invariance. Hence Dudley's BM in $\mathbb{M}$ is defined by $\dot{x}_t$ being the diffusion in $\mathbb{H}^+$ with infinitesimal generator $\frac{\sigma^2}{2}\,\Delta_{\mathbb{H}^+}$ (instead of $\frac{1}{2}\,\Delta_{\mathbb{H}^+}$).

Dudley's definition has been extended by Franchi and Le Jan [3] from the flat Minkowski space-time $\mathbb{M}$ to an arbitrary Lorentzian manifold $\mathcal{M}$. A Franchi–Le Jan Brownian motion in $\mathcal{M}$ is obtained by "rolling without slipping" $\mathbb{M}$ on $\mathcal{M}$ in such a way that $\mathbb{M}$ remains tangent to $\mathcal{M}$, with the Minkowskian structure of the tangent space to $\mathcal{M}$ agreeing with that of $\mathbb{M}$, and the contact point describing a Dudley BM in $\mathbb{M}$; the locus in $\mathcal{M}$ of the contact point is then a Franchi–Le Jan BM in $\mathcal{M}$. More rigorously, a C^1, timelike curve x_t in $\mathcal{M}$, parametrized by its proper time, is a Franchi–Le Jan BM whenever the pull-back $P^{-1}_{x;\,0,t}\dot{x}_t$ of its velocity $\dot{x}_t$ by the parallel transport $P_{x;\,0,t}\colon T_{x_0}\mathcal{M} \to T_{x_t}\mathcal{M}$ along the path $(x_s, s \in [0,t])$ is the velocity of a Dudley BM in the Minkowskian space $T_{x_0}\mathcal{M}$. Equivalently, if $F_t = (e_0(t), e_1(t), \ldots, e_d(t))$ is for each t a random orthonormal Minkowskian frame of $T_{x_t}\mathcal{M}$, depending previsibly upon t, and such that $e_0(t) = \dot{x}_t$, then the Itô differential $\mathrm{d}\dot{x}_t$ (for the horizontal connection) is read in the frame F_t as $(0, \sigma\,\mathrm{d}B^1_t, \ldots, \sigma\,\mathrm{d}B^d_t)$, where $(B^1, \ldots, B^d)$ is a standard BM in $\mathbb{R}^d$.

2. Spacelike covariant processes

In a Lorentzian manifold, do there exist random spacelike *curves having Lorentz covariance?* From a physicist's viewpoint, this question is quite uninteresting; spacelike curves do not seem to have any physical meaning, and are hardly ever mentioned, if at all, in books on relativity. But from a geometric point of view, covariant $\mathcal{M}$-valued spacelike stochastic processes can be considered as mathematical tools for the study of $\mathcal{M}$, on a par with the Franchi–Le Jan timelike motion in $\mathcal{M}$, or with Brownian motion in a Riemannian manifold. We shall henceforth sketch a possible definition of such processes.

A C^1, spacelike curve x is naturally parametrized by its "proper arc length" t, which can be defined by $q(\dot{x}_t, \dot{x}_t) = -1$. (The orientation of the curve is not intrinsically provided by the space-time structure; it has to be chosen.) The instantaneous direction $\dot{x}_t \in T_{x_t}\mathcal{M}$ can be pulled back to $T_{x_0}\mathcal{M}$ by parallel transport. This yields a curve $y_t = P^{-1}_{x;\,0,t}\dot{x}_t$ in the tangent space $T_{x_0}\mathcal{M}$, or rather in the negative-unit ball $\mathbb{H}^{\mathrm{sp}}_{T_{x_0}\mathcal{M}}$ of that space, because $q(y_t, y_t) = -1$. The dynamics of x in $\mathcal{M}$ is fully described by that of y in $\mathbb{H}^{\mathrm{sp}}_{T_{x_0}\mathcal{M}}$, and either of them is intrinsic if and only if so is also the other. Now, $\mathbb{H}^{\mathrm{sp}}_{T_{x_0}\mathcal{M}}$ can be identified with $\mathbb{H}^{\mathrm{sp}}$ by choosing orthonormal bases in $T_{x_0}\mathcal{M}$ and in $\mathbb{M}$. The problem of defining covariant spacelike random curves in $\mathcal{M}$ thus reduces to defining covariant stochastic processes in $\mathbb{H}^{\mathrm{sp}}$. Since the intrinsic structure of $\mathbb{H}^{\mathrm{sp}}$ is that of a Lorentzian submanifold of $\mathbb{M}$, constructing x amounts to constructing an intrinsic stochastic process valued in the Lorentzian manifold $\mathbb{H}^{\mathrm{sp}}$.

We have such a process at hand: the Franchi–Le Jan timelike random motion in the manifold $\mathbb{H}^{\mathrm{sp}}$. This leads to the following definition: *A spacelike random curve x in $\mathcal{M}$, parametrized by arc length, will be called a second-order Brownian motion if its direction $\dot{x}_t$, when parallel-transported back to $T_{x_0}\mathcal{M}$, moves as a Franchi–Le Jan process in the d-dimensional Lorentzian manifold $\mathbb{H}^{\mathrm{sp}}_{T_{x_0}\mathcal{M}}$.*

Naturally, in this setting, the Franchi–Le Jan processes are called *first-order Brownian motions.* Observe that, since the paths of first-order Brownian motions are C^1, those of second-order BM are C^2: regularity has increased by 1, this is the price paid for a spacelike evolution with an intrinsic random noise. On the other hand, first-order BM has stochastic dimension d (its filtration is that of a BM in $\mathbb{R}^d$), and second-order BM has stochastic dimension $d-1$, because one degree of freedom is lost when passing from $\mathcal{M}$, which is $(d+1)$-dimensional, to $\mathbb{H}^{\mathrm{sp}}$, which is d-dimensional.

Remark also that the intrinsic acceleration $\ddot{x}_t$ of a second-order Brownian motion is not only timelike, but also future-pointing; this is a sort of convexity property. Second-order Brownian motions with past-pointing acceleration also exist; a new definition is not necessary, for reversing the time-orientation of $\mathcal{M}$ does the trick.

It may be worth noticing that, even in the special-relativistic case when the manifold $\mathcal{M}$ is the flat Minkowski space-time $\mathbb{M}$, the definition of second-order Brownian motions involves a Franchi–Le Jan process in the *curved* space-time $\mathbb{H}^{\mathrm{sp}}$; the general-relativistic setup cannot be avoided.

There is no reason to stop there, and Brownian motions of higher orders can similarly be defined: *Brownian motion of order n is a spacelike random curve x in $\mathcal{M}$, parametrized by arc-length, such that the motion of $P^{-1}_{x;\,0,t}\dot{x}_t$ in $T_{x_0}\mathcal{M}$ is Brownian of order $n-1$ in the Lorentzian sub-manifold $\mathbb{H}^{\mathrm{sp}}_{T_{x_0}\mathcal{M}}$ of $T_{x_0}\mathcal{M}$*. Brownian motion of order n exists for $n \leqslant d$, is of class C^n, and has stochastic dimension $d+1-n$. Its law depends on the initial conditions

$$x_0 \in \mathcal{M}; \quad \dot{x}_0 \in \mathbb{H}^{\mathrm{sp}}_{T_{x_0}\mathcal{M}} = \mathcal{M}^1; \quad \ddot{x}_0 \in \mathbb{H}^{\mathrm{sp}}_{T_{\dot{x}_0}\mathcal{M}^1} = \mathcal{M}^2; \quad \ldots \quad x_0^{(n)} \in \mathbb{H}^{+}_{T_{x_0^{(n-1)}}\mathcal{M}^{n-1}},$$

where all $\mathbb{H}$-spaces are of the form $\mathbb{H}^{\mathrm{sp}}$, except the last one which is an $\mathbb{H}^{+}$ and whose Riemannian structure allows randomness to enter the picture as an intrinsic Riemannian Brownian motion. The n-th derivative is future-pointing, but can also be made past-pointing, by replacing $\mathbb{H}^{+}$ with $\mathbb{H}^{-}$. When $n=d$, the timelike condition $x_0^{(n)} \in \mathbb{H}^{+}_{T_{x_0^{(n-1)}}\mathcal{M}^{n-1}}$ on the n-th derivative can also be replaced by the spacelike condition $x_0^{(n)} \in \mathbb{H}^{\mathrm{sp}}_{T_{x_0^{(n-1)}}\mathcal{M}^{n-1}}$, because this $\mathbb{H}^{\mathrm{sp}}$, a one-dimensional hyperbola, is no longer Lorentzian but Riemannian, and supports an intrinsic Brownian motion (two of them in fact, one on each branch of the hyperbola).

The stochastic process x_t is not Markov, but $(x_t, \dot{x}_t, \ldots, x_t^{(n)})$ is a Markovian, very degenerate diffusion. In the deterministic case when the parameter σ (which rules the behavior of $x^{(n)}$) vanishes, x simply becomes a curve with null covariant $n+1$-st derivative.

3. The simplest example

To make these definitions less abstract, we shall explicitly describe a second-order Brownian motion x_t valued in the Minkowskian space-time $\mathbb{M} = \mathbb{R} \times \mathbb{E}$ with scalar product $q(x,y) = x^0 y^0 - x^{\rightarrow} \cdot y^{\rightarrow}$. We assume that $d = \dim \mathbb{E} \geqslant 2$. The velocity $y_t = \dot{x}_t$ is a Franchi–Le Jan Brownian motion in the hyperboloid $\mathbb{H}^{\mathrm{sp}} = \{y \in \mathbb{M}\colon\ q(y,y) = -1\}$. The equations governing y_t are the equations of timelike geodesics in $\mathbb{H}^{\mathrm{sp}}$, with an additional noise term on $\mathrm{d}\dot{y}_t$. The geodesic with initial conditions $y_0 \in \mathbb{H}^{\mathrm{sp}}$ and $\dot{y}_0 \in T_{y_0}\mathbb{H}^{\mathrm{sp}}$, where $q(\dot{y}_0, \dot{y}_0) = 1$, is given in $\mathbb{M}$ by $y_t = y_0 \operatorname{ch} t + \dot{y}_0 \operatorname{sh} t$ (see for instance [4]); so its law of motion is

$$\mathrm{d}y_t = \dot{y}_t\,\mathrm{d}t, \quad \mathrm{d}\dot{y}_t = y_t\,\mathrm{d}t.$$

The noise term to be added to $\mathrm{d}\dot{y}_t$ is (σ times) a Brownian differential in the $(d-1)$-dimensional Euclidean space S_t orthogonal to y_t and $\dot{y}_t$ in $\mathbb{M}$. It is convenient to use Stratonovich differentials (we denote them by the symbol δ instead of d), to avoid the drift term which compensates for the extrinsic curvature of $\mathbb{H}^{+}_{T_{y_t}\mathbb{H}^{\mathrm{sp}}}$ in $T_{y_t}\mathbb{H}^{\mathrm{sp}}$. The noise term can for instance be written as $\sigma\ell(\delta B^{\rightarrow})$, where $B^{\rightarrow}$ is an ordinary Brownian motion in the Euclidean space $\mathbb{E}$, and $\ell : \mathbb{E} \to S_t$ any linear map such that $\ell\ell' = \mathrm{Id}_{S_t}$, with $\ell' : S_t \to \mathbb{E}$ standing for the adjoint of ℓ. A possible choice for ℓ is as follows. Define

$$r_t = \|y_t^{\rightarrow} \wedge \dot{y}_t^{\rightarrow}\|^2 = (\dot{y}_t^0)^2 - (y_t^0)^2 - 1, \quad u_t^{\rightarrow} = \dot{y}_t^0 y_t^{\rightarrow} - y_t^0 \dot{y}_t^{\rightarrow} \quad \text{and} \quad v_t^{\rightarrow} = \dot{y}_t^0 \dot{y}_t^{\rightarrow} - y_t^0 y_t^{\rightarrow};$$

observe that $u_t^{\rightarrow} \cdot u_t^{\rightarrow} = r_t + 1$, $u_t^{\rightarrow} \cdot v_t^{\rightarrow} = 0$ and $v_t^{\rightarrow} \cdot v_t^{\rightarrow} = r_t(r_t+1)$. In $\mathbb{M}$, the three vectors y_t, $\dot{y}_t$ and $z_t = (r_t, v_t^{\rightarrow})$ remain orthogonal to each other, with $q(z_t, z_t) = -r_t$. We shall define ℓ so that $\ell(y_t^{\rightarrow})$ and $\ell(\dot{y}_t^{\rightarrow})$ are multiples of z_t, and that ℓ maps isometrically the space orthogonal to $y_t^{\rightarrow}$ and $\dot{y}_t^{\rightarrow}$ in $\mathbb{E}$ onto the space orthogonal to z_t in S_t. This can be achieved by putting

$$\ell(u_t^{\rightarrow}) = 0, \quad \ell(v_t^{\rightarrow}) = \sqrt{r_t + 1}\, z_t, \quad \ell(w^{\rightarrow}) = (0, w^{\rightarrow}) \quad \text{if } w^{\rightarrow} \text{ is orthogonal to } u_t^{\rightarrow} \text{ and } v_t^{\rightarrow}.$$

Finally, a system of Stratonovich equations in $\mathbb{M}$ for the Franchi–Le Jan Brownian motion y_t in $\mathbb{H}^{\mathrm{sp}}$ is

$$\begin{cases} \delta y_t^0 = \dot{y}_t^0 \delta t, \\ \delta y_t^{\rightarrow} = \dot{y}_t^{\rightarrow} \delta t, \\ \delta \dot{y}_t^0 = y_t^0 \delta t + \sigma \dfrac{v_t^{\rightarrow} \cdot \delta B_t^{\rightarrow}}{\sqrt{r_t + 1}}, \\ \delta \dot{y}_t^{\rightarrow} = y_t^{\rightarrow} \delta t + \sigma \left[\delta B_t^{\rightarrow} - \dfrac{u_t^{\rightarrow} \cdot \delta B_t^{\rightarrow}}{r_t + 1} u_t^{\rightarrow} + (\sqrt{r_t + 1} - 1) \dfrac{v_t^{\rightarrow} \cdot \delta B_t^{\rightarrow}}{r_t(r_t + 1)} v_t^{\rightarrow} \right], \end{cases}$$

with initial conditions y_0 and $\dot{y}_0$ in $\mathbb{M}$ such that $q(y_0, y_0) = -1$, $q(y_0, \dot{y}_0) = 0$ and $q(\dot{y}_0, \dot{y}_0) = 1$. The second-order Brownian motion $x_t = (x_t^0, x_t^{\rightarrow})$ in $\mathbb{M}$ is now obtained by complementing this system with the additional two equations

$$\begin{cases} \delta x_t^0 = y_t^0 \delta t, \\ \delta x_t^{\rightarrow} = y_t^{\rightarrow} \delta t \end{cases}$$

and an initial condition $x_0 \in \mathbb{M}$.

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