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A Striktpositivstellensatz for measurable functions ☆

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Abstract

A weighted sums of squares decomposition of positive Borel measurable functions on a bounded Borel subset of the Euclidean space is obtained via duality from the spectral theorem for tuples of commuting self-adjoint operators. The analogous result for polynomials or certain rational functions was amply exploited during the last decade in a variety of applications. ***To cite this article: M. Putinar, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Un Striktpositivstellensatz pour les fonctions mesurables. La décomposition dans une somme de carrés ponderés d'une fonction de Borel positive sur un ensemble mesurable est obtenue grace au théorème spectral pour les systemes commutatifs des opérateurs autoadjoints. Un résultat similaire, obtenu pour les pôlynomes ou certaines fonctions rationelles a été fortement exploité au cours des dernières douze années pour l'optimisation non-lineaire et non-convexe. ***Pour citer cet article : M. Putinar, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

To put our main result into the current real algebra context, we recall below the abstract framework for studying linear decompositions into weighted sums of squares.

Let A be a commutative algebra with 1, over the rational field. A *quadratic module* $Q \subset A$ is a subset of A such that $Q + Q \subset Q$, $1 \in Q$ and $a^2 Q \subset A$ for all $a \in A$. We denote by $Q(F; A)$ or simply $Q(F)$ the quadratic module generated in A by the set F. That is $Q(F; A)$ is the smallest subset of A which is closed under addition and multiplication by squares $a^2,\ a \in A$, containing M and the unit $1 \in A$. If F is finite, we say that the quadratic module is finitely generated. A quadratic module which is also closed under multiplication is called a *quadratic preordering*. A quadratic module Q is called archimedean if the constant function 1 belong to its algebraic interior, that is, for every $f \in Q$ there exists $\epsilon > 0$ such that $1 + tf \in Q$ for all rational numbers t satisfying $0 \leqslant t \leqslant \epsilon$.

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Assume that $A = \mathbb{R}[x_1, \ldots, x_d]$ is the polynomial algebra. The positivity set $P(Q)$ of $Q \subset \mathbb{R}[x_1, \ldots, x_d]$ is the set of all points $x \in \mathbb{R}^d$ for which $q(x) \geqslant 0$, $q \in Q$.

The following Striktpositivstellensatz has attracted in the last decade a lot of attention from practitioners of polynomial optimization: *Let $Q \subset \mathbb{R}[x_1, \ldots, x_d]$ be an archimedean quadratic module and assume that a polynomial f is positive on $P(Q)$. Then $f \in Q$.*

This fact was discovered by the author [10], generalizing a series of similar decomposition theorems, proved for preorderings instead of quadratic modules by Stone, Krivine, Handelman and others. As a culmination of these results, Schmüdgen proved in [14] that the compactness of $P(Q)$ for finitely generated Q implies the archimedean property of the preorder Q.

In plain language, the above result can be stated as follows. Denote by Σ^2 the convex cone of sums of squares in the polynomials ring $\mathbb{R}[x_1, \ldots, x_d]$. Let $p_1, \ldots, p_k \in \mathbb{R}[x_1, \ldots, x_d]$ be polynomials, so that the quadratic module generated by them $Q(p_1, \ldots, p_k) = \Sigma^2 + p_1\Sigma^2 + \cdots + p_k\Sigma^2$ is archimedean. That is, there exists $\epsilon > 0$ so that $1 - \epsilon(x_1^2 + \cdots + x_d^2) \in Q$ (for the reduction of Q archimedean to this criterion see [9]). The stated Striktpositivstellensatz asserts: if a polynomial P is positive on the set $S(p_1, \ldots, p_k) = \{x = (x_1, \ldots, x_d);\ p_i(x) \geqslant 0,\ 1 \leqslant i \leqslant k\}$, then $P \in Q(p_1, \ldots, p_k)$.

A simple duality argument implies under the above conditions that every linear functional $L \in \mathbb{R}[x]'$ which is non-negative on the quadratic module $Q(p_1, \ldots, p_k)$ is represented by a positive Borel measure μ, supported by the basic semi-algebraic set $S(p_1, \ldots, p_k)$:

$$L(f) = \int\limits_{S(p_1,\ldots,p_k)} f\,\mathrm{d}\mu, \quad f \in \mathbb{R}[x].$$

The correspondence between the above Positivstellensatz and the multivariate moment problem with prescribed compact semi-algebraic supports works fruitfully in both directions. First, the original proof of the Positivstellensatz was obtained via the standard moment problem solution offered by the spectral theorem (see [10]; and [9] for an algebraic proof). Second, and more important for applications, it was J.B. Lasserre [7,8] who has interpreted the moments

$$y_\alpha = \int x^\alpha\,\mathrm{d}\mu, \quad \alpha \in \mathbb{N}^d,$$

of the representing measure as new independent variables and has obtained a hierarchy of linear, semi-definite optimization problem converging to the minimization of a given polynomial on a compact semi-algebraic set. For more details towards applications and theoretical ramifications see [1–3,11].

2. Main result

The aim of this note is to prove a natural extension of the polynomial Positivstellensatz to algebras of Borel measurable functions defined on Euclidean space. Although a more general statement, on an arbitrary locally compact space or even on a non-commutative C^*-algebra is possible to deduce with similar techniques, we consider that the Euclidean space setting is: first, the most important for applications, and second, it contains a specific feature which makes it worth a separate discussion.

Theorem 2.1. *Let Q be a countably generated archimedean quadratic module contained in the algebra $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_d, h_1, \ldots, h_m]$ spanned by the coordinate functions and by Borel measurable functions $h_1, \ldots, h_m$ on $\mathbb{R}^d$. If a function $f \in \mathcal{A}$ is positive on $P(Q)$, then $f \in Q$.*

Proof. Since Q is archimedean, there exists $\epsilon > 0$ such that $1 - \epsilon(x_1^2 + \cdots + x_d^2 + h_1^2 + \cdots + h_m^2) \in Q$. Thus the positivity set $P(Q)$ is contained in the ball $x_1^2 + \cdots + x_d^2 \leqslant 1/\epsilon$. Because Q is countably generated, the set $P(Q)$ is Borel measurable.

The fact that Q is archimedean as a convex cone, means that for every $h \in \mathcal{A}$ there exists positive constants c, C with the property $C - h, h - c \in Q$. Assume by contradiction that the function f does not belong to Q. According to Marcel Riesz extension of positive functionals [12] (known and rediscovered over the years by many authors [4,6,5]), there exists $L \in \mathcal{A}'$ so that:

$$L(f) \leqslant 0 \leqslant L(q), \quad L(1) > 0,\ q \in Q.$$

Next we use Gelfand–Naimark–Segal construction of a Hilbert space realization of the functional L. Specifically, $L(g^2) \geqslant 0$ for all $g \in \mathcal{A}$, and Cauchy–Schwarz' inequality proves that the set $\mathcal{N}$ of functions $h \in \mathcal{A}$, $L(h) = 0$, is an ideal; whence we can introduce on the quotient algebra $\mathcal{A}/\mathcal{N}$ the positive definite inner product

$$\langle g_1, g_2 \rangle = L(g_1 g_2), \quad g_1, g_2 \in \mathcal{A}/\mathcal{N}.$$

Let $\mathcal{H}$ be the Hilbert space completion of $\mathcal{A}/\mathcal{N} \otimes_{\mathbb{R}} \mathbb{C}$. Since Q is archimedean, the multiplication operators by each generator $x_1, \ldots, x_d, h_1, \ldots, h_m$ extends by linearity to a tuple of commuting *bounded* self-adjoint operators on $\mathcal{H}$, denoted by $(X, H) = (X_1, \ldots, X_d, H_1, \ldots, H_m)$, respectively. In view of the Spectral Theorem [13], there exists a positive measure σ on $\mathbb{R}^{d+m}$, so that, for all bounded Borel functions $F(x_1, \ldots, x_d, y_1, \ldots, y_m)$ we have

$$\big\langle F(X_1, \ldots, X_d, H_1, \ldots, H_m)1, 1\big\rangle = \int\limits_{\mathbb{R}^{d+m}} F \, \mathrm{d}\sigma.$$

From here we deduce that $H_j = h_j(X_1, \ldots, X_d)$, $1 \leqslant j \leqslant m$. Indeed, let $p, q \in \mathcal{A}$ and fix the index j, $1 \leqslant j \leqslant m$. Then by its very definition the operator H_j satisfies

$$\langle H_j p, q \rangle = \big\langle h_j(x) p, q \big\rangle = \big\langle h_j(X) p, q \big\rangle,$$

where $h_j(X)$ is the Borel functional calculus applied to the commuting tuple of self-adjoint operators X, see for details [13]. Since the equivalence classes of the elements p, q span a dense subset of the Hilbert space $\mathcal{H}$, we infer $H_j = h_j(X)$.

Therefore the measure σ is the push forward of a positive measure on $\mathbb{R}^d$ by the graph map $x \mapsto (x, h_1(x), \ldots, h_m(x))$, $x \in \mathbb{R}^d$:

$$\big\langle F(X_1, \ldots, X_d, H_1, \ldots, H_m)1, 1\big\rangle = \int\limits_{\mathbb{R}^d} F\big(x, h_1(x), \ldots, h_m(x)\big) \, \mathrm{d}\mu(x).$$

This shows that the measure μ has compact support, contained in the ball centered at 0, of radius $1/\epsilon$.

Let $r \in Q$ be a generator of Q. Regarded as a Borel function of the variables $x \in \mathbb{R}^d$, the element r satisfies for all polynomials $p \in \mathbb{R}[x]$:

$$0 \leqslant L\big(rp^2\big) = \big\langle r(X, H)p, p\big\rangle = \int rp^2 \, \mathrm{d}\mu.$$

By Stone–Weierstrass Theorem, the same positivity is inherited from polynomials $p(x)$ to all continuous functions, and by the σ-additivity of the measure μ, to all bounded Borel measurable functions. In particular, for a characteristic function of a measurable set A, $\chi_A = \chi_A^2$, we obtain $0 \leqslant \int r \chi_A \, \mathrm{d}\mu$. In other terms,

$$\mu\big\{x \in \mathbb{R}^d;\ r\big(x, h_1(x), \ldots, h_m(x)\big) < 0\big\} = 0.$$

Since

$$P(Q) = \Cap_{r_n} \big\{x \in \mathbb{R}^d;\ r_n\big(x, h_1(x), \ldots, h_m(x)\big) \geqslant 0\big\},$$

where r_n is an at most countable system of generators for Q we find $\mu(\mathbb{R}^d \setminus P(Q)) = 0$.

Finally, recall from the statement that $f|_{P(Q)} > 0$. On the other hand, by the construction of the functional we have

$$\int\limits_{P(Q)} f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu = \big\langle f(X, H)1, 1\big\rangle = L(f) \leqslant 0.$$

But $\mu(\mathbb{R}^d) > 0$, and thus we reach a contradiction. □

The reader will encounter no complications in specializing the theorem above and its proof to a finitely generated quadratic module. We simply state the result.

Corollary 2.2. *Let $q_1, \ldots, q_n$ be elements of the algebra $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_d, h_1, \ldots, h_m]$ generated by the coordinate functions and by Borel measurable functions $h_1, \ldots, h_m$ on $\mathbb{R}^d$. Let $\Sigma\mathcal{A}^2$ denote the convex cone of sums of squares, and consider the Borel measurable set*

$$P(q_0, q_1, \ldots, q_n) = \{x \in \mathbb{R}^d;\ q_i(x) \geqslant 0,\ 0 \leqslant i \leqslant n\},$$

where $q_0(x) = 1 - (x_1^2 + \cdots + x_d^2 + h_1^2 + \cdots + h_m^2)$.

If a function $f \in \mathcal{A}$ *is positive on* $P(q_0, q_1, \ldots, q_n)$*, then* $f \in \Sigma\mathcal{A}^2 + q_0\Sigma\mathcal{A}^2 + \cdots + q_n\Sigma\mathcal{A}^2$.

When trying to extend Lasserre's linearization procedure to this new framework, the mixed moments

$$y_{\alpha,\beta} = \int x^\alpha h(x)^\beta \, \mathrm{d}\mu, \quad \alpha \in \mathbb{N}^d,\ \beta \in \mathbb{N}^m,$$

should be considered, together will all *algebraic* dependence relations among the functions $x_1, \ldots, x_d, h_1(x), \ldots, h_m(x)$. For instance it may happen that $h_i(x)$ is a characteristic function of a Borel set, in which case $h_i^2 = h_i$, or that $h(x)$ is an m-tuple of algebraic functions, in which case a polynomial dependence $P(x, h(x)) = 0$ holds.

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