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Holomorphic extension of fundamental solutions of elliptic linear partial differential operators of the second order with analytic coefficients

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Abstract

We prove that every fundamental solution of an elliptic linear partial differential operator of the second order with analytic coefficients and simple complex characteristics in an open set $\Omega \subset \mathbb{R}^n$ can be continued at least locally as a multi-valued analytic function in $\mathbb{C}^n$ up to the complex bicharacteristic conoid. This extension ramifies or not along its singular set the bicharacteristic conoid and belongs to the Nilsson class. ***To cite this article: S. Lukasiewicz, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Prolongement holomorphe de solutions élémentaires d'opérateurs différentiel elliptiques du second degré à coefficients analytiques. Toute solution élémentaire d'un opérateur différentiel elliptique du second ordre à coefficients analytiques et à caractéristiques complexes simples peut se prolonger holomorphiquement (au moins localement) dans $\mathbb{C}^n$ jusqu'au cône bicaractéristique complexe en se ramifiant éventuellement. Cette extension appartient à la classe de Nilsson. ***Pour citer cet article : S. Lukasiewicz, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction, notations and definitions

This kind of extension problems were studied for fundamental solutions of certain classes of hyperbolic differential operators in classical works of J. Leray [4]. We saw that we could generalize and adapt some of his results on hyperbolic operators to the elliptic case. In [1], L. Boutet de Monvel proved that the singular fundamental solution of an elliptic fundamental solution of an elliptic homogeneous linear differential operator with constant coefficients and simple characteristics is a solution of a regular holonomic $\mathcal{D}$-module. When the coefficients are analytic and the symbol of the operator has constant coefficients D. Meyer [5] has proved that the singular fundamental solution of such a differential operator is still a solution of a regular holonomic $\mathcal{D}$-module. In both cases, as a consequence, the fundamental solutions of these operators extend up to the bicharacteristic conoid and belong to the Nilsson class.

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Our result for elliptic linear partial differential operators with analytic coefficients generalize this consequence for the second order. We use the standard multi-index notation. Let $\Omega \subset \mathbb{R}^n$ be an open set, we denote by $\mathcal{D}'(\Omega)$ the space of distributions on Ω and by $P(x,D) = \sum_{|\alpha| \leqslant 2} a_\alpha(x) \frac{\partial}{\partial x^\alpha}$ a linear differential operator of the second order with analytic coefficients in Ω. Let $T^*\Omega$ be the cotangent bundle of Ω and let $\xi_1, \ldots, \xi_n$ be a coordinate system on its fiber at $(x_1, \ldots, x_n)$. The principal symbol of P is defined as usual. P is said to be elliptic if $a(x,\xi) \neq 0$ in $T^*\Omega \setminus 0$ where 0 denotes the zero section of $T^*\Omega$.

In what follows, the canonical projection of the usual bicharacteristics on Ω will also be called the bicharacteristics of P. An elliptic operator has no real bicharacteristics. However, if we consider, following Leray [3], the characteristic variety as a complex variety in $T^*\mathbb{C}^n$, one defines complex bicharacteristics associated to P: the complex bicharacteristics will be solutions included in the complex characteristic variety of the Hamiltonian system ($t \in \mathbb{C}$): $\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial a(x,\xi)}{\partial \xi_i}$; $\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = -\frac{\partial a(x,\xi)}{\partial x_i}$ with initial datas $x_i(0) = y_i$, $\xi(0) = \xi_i$, $i \in \{1, \ldots, n\}$. The bicharacteristic conoid Γ_y is defined as the union of all complex bicharacteristics with initial data $y \in \mathbb{C}^n$.

Let X be a connected complex analytic manifold, $x \in X$ and f be a an analytic germ at x. A multi-valued analytic function f on X has finite determination if the complex vector space generated by the local branches of f has finite dimension. Let f be a multi-valued analytic function in $\mathbb{C}^n \setminus \Gamma$ where Γ is a complex analytic hypersurface in $\mathbb{C}^n$. We say that f belongs to the Nilsson class [6,7] if f has finite determination and f has moderate growth along Γ: for any open set U in $\mathbb{C}^n$ such that $\Gamma \cap U = \{g(z) = 0\}$, where g is analytic function, there exists a positive integer N such that for every semi-analytic set P, simply connected and relatively compact in $U \setminus \Gamma$, and for every local branch of f, there exists a constant C such that $\forall z \in P$, $|f(z)| \leqslant \frac{C}{|g(z)|^N}$.

In this Note, a distribution $E(x,y) \in \mathcal{D}'(\Omega \times \Omega)$ depending on y as a parameter is called a fundamental solution of P if $PE(x,y) = \delta(x-y)$.

2. Preliminary study on a few examples

We first notice that every fundamental solution of an elliptic operator P is the sum of a fundamental solution E such that its singular support is reduced to a point $y \in \mathbb{R}^n$ (the singular solution) and an analytic solution of $Pu = 0$.

2.1. The Laplacian in $\mathbb{R}^n$

A fundamental solution of the Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^n$ is for $n \geqslant 3$: $E(x,y) = \frac{c_n}{(2-n)|x-y|^{n-2}}$ where c_n denotes the area of the $(n-1)$-dimensional real unit sphere in $\mathbb{R}^n$ and $|x-y| = [\sum_{i=1}^n (x_i - y_i)^2]^{\frac{1}{2}}$. For $n = 2$, $E(x,y) = \frac{\log|x-y|}{2\pi}$. So $E(x,y)$ has a holomorphic extension to $\mathbb{C}^n \setminus Q$ where Q is the isotropic cone of $\mathbb{C}^n$: $\sum_{i=1}^n (z_i - z_i')^2 = 0$. Since $|x-y|^{n-2} = [\sum_{i=1}^n (x_i - y_i)^2]^{\frac{n-2}{2}}$, $E(x,y)$ belongs to the Nilsson class. In particular, we stress the fact that the isotropic cone is indeed the union of all complex bicharacteristics with initial data y. We can too describe precisely the ramification: for $n = 2$, $E(x,y)$ has a logarithmic ramification along the isotropic cone; for n even, $n \geqslant 4$, $E(x,y)$ is meromorphic with poles on the isotropic cone and, last case, for n odd, $E(x,y)$ has two ramifications along the isotropic cone.

2.2. The generalized Helmholtz operator in $\mathbb{R}^n$

The generalized Helmholtz operator is the operator $\Delta + V(x)$ where $V(x)$ is an analytic function. It is known that the singular fundamental solution has the form: $E(x,y) = \frac{F(x,y)}{|x-y|^{n-2}} + G(x,y)\log|x-y|$ where F, G are analytical functions. So it extends up to the isotropic cone of $\mathbb{C}^n$ and belongs to the Nilsson class.

3. Results

Theorem 3.1. *Let*

$$P(x,D) = \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} + c(x)$$

be an elliptic operator where $a_{i,j}(x)$, $b_j(x)$, $c(x)$ are analytic functions in Ω and $a_{i,j}(x)$ is assumed to be symmetric without loss of generality. Denote by $A_{i,j}(x)$ the inverse matrix of $[a_{i,j}(x)]$ and $\mathrm{d}s$ the Riemannian metric defined by: $\mathrm{d}s^2 = \sum_{i,j=1}^{n} A_{i,j}(x)\,\mathrm{d}x_i\,\mathrm{d}x_j$, then a local fundamental solution of P is given by:

$$E(x,y) = \frac{F(x,y)}{d(x,y)^{n-2}} + G(x,y)\log d(x,y)^2 + H(x,y)$$

where F, G, H denote three analytic functions in a neighborhood of the diagonal $\Omega \times \Omega$.

If the dimension n is odd, we have $G = 0$. There is no logarithmic term and $E(x,y)$ is ramified of order 2. If n is even, $E(x,y)$ has a polar singularity and possibly a logarithmic term.

Corollary 3.2. *With the same notations, every fundamental solutions of an analytic elliptic differential operator of the second order P can be extended at least locally as a multi-valued analytic function up to $\mathbb{C}^n \setminus \Gamma_y$ where Γ_y is the bicharacteristic conoid with initial data $y \in \Omega$. All those fundamental solutions belong to the Nilsson class.*

Indeed, $d(x,y)^2$ is defined and analytic in a neighborhood of $x = y$. So $d(x,y)^2$ has a holomorphic extension in a neighborhood of $x = y$. Now $d(x,y)^{n-2} = [d(x,y)^2]^{\frac{n-2}{2}}$, so $d(x,y)^{n-2}$ extends to a multi-valued analytic function with at most two ramifications along the isotropic cone $d^2 = 0$. According to the lemma, we have proved the proposition.

Remarks.

(1) Of course, this is true also for such operators on an analytic manifold X.
(2) When the operator has constant coefficients, the extension is global.
(3) The author is grateful to the referee for pointing out that the corollary remains true for elliptic differential operators with complex coefficients and simple characteristics. It is not clear if our proof works by using the Goursat problem (for the even dimension). But we may write instead of it $G = \sum_{k=0}^{\infty} G_k(x,y)d^{2k}$, take G_0 as in the proof and get the same kind of sequence of differential equations used in the proof; this asymptotics expansion is (with the same argument) convergent for d^{2k} small enough.

Proof of the theorem. The proof of the theorem is similar to the one given in Hadamard [2] for hyperbolic operators. Let us sketch the proof. First, assume n is odd and that $E(x,y)d^{n-2}$ expands in integral powers of d^2: $E(x,y) = \frac{1}{d^{n-2}} \sum_{k=0}^{\infty} F_k(x,y)d^{2k}$.

We compute $P(E(x,y))$ and we use the fact that $P(E(x,y)) = 0$ when $x \neq y$. Reparametrizing in geodesic normal coordinates with a parameter s, we get the following crucial sequence of differential equations:

$$s\frac{\mathrm{d}F_0}{\mathrm{d}s} + \big(\theta(x,y) - m - 1\big)F_0 = 0; \qquad \forall k \geqslant 1 \quad s\frac{\mathrm{d}F_k}{\mathrm{d}s} + \big(\theta(x,y) - k - m - 1\big)F_k = -\frac{1}{4(k-m)}P[F_{k-1}]$$

where $m = \frac{n-2}{2}$ and

$$\theta(x,y) = \frac{1}{4}\left(\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2(d^2)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial(d^2)}{\partial x_j}\right)$$

is an analytic function of x and y. The system is solvable if and only if $\theta(y,y) = m + 1 = \frac{n}{2}$; this is indeed the case as one sees from a Taylor expansion of d^2 in the neighborhood of (y,y) and we may integrate it to get $F_0, F_1, \ldots, F_k, \ldots$. We then check by a technique similar to Cauchy–Kowalewska theorem due to Hadamard [2] that the asymptotic expansion $\sum_{k=0}^{\infty} F_k(x,y)d^{2k}$ is convergent for d^{2k} small enough.

For n even, with the same sequence of differential equations than for n odd, we may determine only $F_0, \ldots, F_{m-1}$. We expand then $E(x,y)$ in the following way: $E = \sum_{k=0}^{m-1} F_k(x,y)d^{2k} + G\log d^2 + H$. Since the logarithmic term must cancel out , $P(G) = 0$. So we see that:

$$s\frac{\mathrm{d}G}{\mathrm{d}s} + \big(\theta(x,y) - 1\big)G = -\frac{1}{4}P[F_{m-1}]$$

on the characteristic conoid $d^2 = 0$. So we can compute the value G_0 of G on the characteristic conoid:

$$G_0(s) = -\frac{F_0(s)}{4s^m} \int_0^s \frac{P(F_{m-1})\xi^{m-1}}{F_0(\xi)} \, \mathrm{d}\xi.$$

Thus we have a Goursat problem and this determines G in a unique way as proved in Hadamard [2] paragraph 64. Finally, H is chosen such that it satisfies the analytic partial differential $P(H) = 0$ in a neighborhood of y.

We finally check with Green's formula help that this asymptotic solution is indeed a fundamental solution. □

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