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Complex Analysis

On boundary angular derivatives of an analytic self-map of the unit disk

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Abstract

Given complex numbers $s_0, \dots, s_N$, we present necessary and sufficient conditions for the existence of a function f analytic and bounded by one in modulus on the open unit disk which admits the nontangential boundary asymptotic expansion $f(z) = s_0 + s_1(z - t_0) + \cdots + s_N(z - t_0)^N + \mathrm{o}((z - t_0)^N)$ at a given point t_0 on the unit circle. This criterion can be considered as a boundary analog of the classical result of I. Schur. ***To cite this article: V. Bolotnikov, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Un problème aux limites à dérivées non normales pour une application analytique du disque sur lui-même. On se donne des nombres complexes $s_0, \dots, s_N$, et on établit des conditions nécessaires et suffisantes d'existence d'une fonction analytique, définie sur le disque unité ouvert, bornée en module par un et admettant un développement asymptotique non tangentiel, en un point t_0 du bord, du type $f(z) = s_0 + s_1(z - t_0) + \cdots + s_N(z - t_0)^N + \mathrm{o}((z - t_0)^N)$. Ce critère peut être considéré commec l'analogue à la frontière du résultat classique de I. Schur. ***Pour citer cet article : V. Bolotnikov, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let $\mathcal{S}$ denote the Schur class of analytic functions mapping the unit disk $\mathbb{D}$ into its closure (i.e., the closed unit ball of H^∞). Characterization of Schur class functions in terms of their Taylor coefficients goes back to I. Schur [9] (and to C. Carathéodory [6] for a related class of functions):

Theorem 1.1. *There is a function $f(z) = s_0 + s_1 z + \cdots + s_{n-1} z^{n-1} + \cdots \in \mathcal{S}$ if and only if the lower triangular Toeplitz matrix $\mathbb{U}_n$ (see formula (7) below) is a contraction, i.e., if and only if the matrix $\mathbb{P} = I_n - \mathbb{U}_n \mathbb{U}_n^*$ is positive semidefinite.*

By conformal change in variable, a similar result is established for an arbitrary point $\zeta \in \mathbb{D}$ at which the Taylor coefficients are prescribed: there exists a function $f \in \mathcal{S}$ of the form,

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$$f(z) = s_0 + s_1(z-\zeta) + \cdots + s_{n-1}(z-\zeta)^{n-1} + \cdots, \tag{1}$$

if and only if certain matrix $\mathbb{P}$ (explicitly constructed in terms of ζ and $s_0, \dots, s_n$) is positive semidefinite. Furthermore, if $\mathbb{P}$ is positive definite, then there are infinitely many Schur class functions f of the form (1). If $\mathbb{P} \geqslant 0$ is singular, then there is a unique function $\in \mathcal{S}$ of the form (1) and this unique function is a finite Blaschke product of degree equal to the rank of $\mathbb{P}$.

Let us consider a similar question in the "boundary" setting, when the Taylor expansion (1) at $\zeta \in \mathbb{D}$ is replaced by the asymptotic expansion at some point t_0 on the unit circle $\mathbb{T}$.

Question. Given a point $t_0 \in \mathbb{T}$ and given numbers $s_0, \dots, s_n \in \mathbb{C}$, does there exist a function $f \in \mathcal{S}$ which admits the asymptotic expansion:

$$f(z) = s_0 + s_1(z-t_0) + \cdots + s_N(z-t_0)^N + \mathrm{o}\big((z-t_0)^N\big) \tag{2}$$

as z tends to t_0 nontangentially?

In what follows, all the limits are nontangential. Observe that asymptotic equality (2) is equivalent to the existence of the following boundary limits $f_j(t_0)$ and equalities:

$$f_j(t_0) := \lim_{z \to t_0} \frac{f^{(j)}(z)}{j!} = s_j \quad \text{for } j = 1, \dots, N. \tag{3}$$

We will denote by $\mathbf{BP}_N$ the interpolation problem which consists of finding a function $f \in \mathcal{S}$ satisfying boundary interpolation conditions (3).

Proposition 1.2. *Given $t_0 \in \mathbb{T}$ and $s_0, \dots, s_N \in \mathbb{C}$, condition $|s_0| \leqslant 1$ is necessary and condition $|s_0| < 1$ is sufficient for the problem $\mathbf{P}_N$ to have a solution.*

The necessity of condition $|s_0| \leqslant 1$ follows from the very definition of the class $\mathcal{S}$. On the other hand, if $|s_0| < 1$, then there are infinitely many functions $f \in \mathcal{S}$ satisfying (3); see [3, Theorem 1.2] for the proof.

The objective of this paper is to present *necessary and sufficient conditions* for the existence of an $f \in \mathcal{S}$ satisfying (3). Skipping the trivial case $N = 0$ (where condition $|s_0| \leqslant 1$ is necessary and sufficient for the problem $\mathbf{BP}_0$ to have a solution), we review the case $N = 1$ which is mostly known as the sketch of the proof below shows.

Theorem 1.3. *There exists a function $f \in \mathcal{S}$ such that*

$$f(z) = s_0 + s_1(z-t_0) + \mathrm{o}\big((z-t_0)\big) \quad \textit{as } z \to t_0, \tag{4}$$

if and only if either $\quad$ (1) $|s_0| < 1$ $\quad$ *or* $\quad$ (2) $|s_0| = 1$ $\quad$ *and* $\quad t_0 s_1 \bar{s}_0 \geqslant 0$.

Proof. Condition $|s_0| \leqslant 1$ is necessary. If $|s_0| = 1$, then the necessity of condition $t_0 s_1 \bar{s}_0 \geqslant 0$ follows from the Carathéodory–Julia theorem [7]. Conversely, if $|s_0| < 1$, then there are infinitely many functions $f \in \mathcal{S}$ of the form (4) by Proposition 1.2. On the other hand, if the necessary conditions (2) are satisfied, then a straightforward computation shows that the rational function,

$$f(z) = s_0 + \frac{(z-t_0)s_1(1-\bar{s}_0\gamma)}{1-(z-t_0)s_1\bar{s}_0 - z\bar{t}_0\bar{s}_0\gamma},$$

belongs to $\mathcal{S}$ and satisfies (4) for every $\gamma \in \mathbb{D}$. The latter formula produces infinitely many solutions to the problem $\mathbf{P}_N$, unless $s_1 = 0$ in which case there is only one solution $f \equiv s_0$. $\square$

To our best knowledge the case $N = 2$ (covered by the next theorem) has not been considered before.

Theorem 1.4. *There exists a unique function $f \in \mathcal{S}$ such that*

$$f(z) = s_0 + s_1(z-t_0) + s_2(z-t_0)^2 + \mathrm{o}\big((z-t_0)^2\big) \quad \textit{as } z \to t_0, \tag{5}$$

if and only if $|s_0| = 1$ and $s_1 = s_2 = 0$. There are infinitely many such functions if and only if either $|s_0| < 1$ or

$$|s_0| = 1, \quad s_1 t_0 \bar{s}_0 > 0 \quad \textit{and} \quad 2\,\mathrm{Re}\big(t_0^2 \bar{s}_0 s_2\big) \geqslant |s_1|^2 - t_0 \bar{s}_0 s_1. \tag{6}$$

A new point here is the following: the two first conditions in (6) guarantee the existence of functions $f \in \mathcal{S}$ satisfying (4). Then for some of these functions to satisfy the extended asymptotic equality (5), it is necessary and sufficient that s_2 belongs to the half-plane determined in terms of s_0 and s_1 by the third condition in (5).

2. The main result

To establish the existence criterion for $N \geqslant 2$, it remains (due to Proposition 1.2) to describe all tuples $\{s_0, s_1, \dots, s_N\}$ with $|s_0| = 1$ satisfying (3) for some $f \in \mathcal{S}$. To present the main result, we first introduce some needed notation. Given $t_0 \in \mathbb{T}$ and $s_0, s_1, \dots, s_N$, we define the lower triangular Toeplitz matrix $\mathbb{U}_n$ and the Hankel matrix $\mathbb{H}_n$ by

$$\mathbb{U}_n = \begin{bmatrix} s_0 & 0 & \cdots & 0 \\ s_1 & s_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ s_{n-1} & \cdots & s_1 & s_0 \end{bmatrix} \quad \text{and} \quad \mathbb{H}_n = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{bmatrix} \tag{7}$$

for every appropriate integer $n \geqslant 1$ (i.e., for $n \leqslant N-1$ in the first formula and for $n \leqslant (N+1)/2$ in the second). We also introduce the upper triangular matrix $\Psi_n(t_0) = [\Psi_{j\ell}]_{j,\ell=1}^n$ with the entries:

$$\Psi_{j\ell} = \begin{cases} 0, & \text{if } j > \ell, \\ (-1)^{\ell-1}\binom{\ell-1}{j-1} t_0^{\ell+j-1}, & \text{if } j \leqslant \ell, \end{cases} \quad \text{for } j, \ell = 1, \dots, n \tag{8}$$

(for example, $\Psi_1(t_0) = t_0$ and $\Psi_2(t_0) = \left[\begin{smallmatrix} t_0 & -t_0^2 \\ 0 & -t_0^3 \end{smallmatrix}\right]$) and finally, the structured matrix,

$$\mathbb{P}_n = [p_{ij}]_{i,j=1}^n = \mathbb{H}_n^{\mathbf{s}} \Psi_n(t_0) \mathbb{U}_n^* \quad \text{with the entries } p_{ij} = \sum_{r=1}^{j} \left(\sum_{\ell=1}^{r} s_{i+\ell-1} \Psi_{\ell r} \right) \bar{s}_{j-r}. \tag{9}$$

The second formula in (9) follows from the first and from (7), (8). Observe that this formula defines the numbers p_{ij} in terms of $\mathbf{s} = \{s_0, \dots, s_N\}$ for every pair (i, j) of indices subject to $i + j \leqslant N + 1$. In particular, if $n \leqslant N/2$, one can define via the second formula in (9) all the entries in the matrix $\mathbb{P}_{n+1}$ except for the rightmost diagonal entry (if $N = 2n$). Let $B_n \in \mathbb{C}^n$ be the vector defined by:

$$B_n := \begin{bmatrix} p_{1,n+1} \\ \vdots \\ p_{n,n+1} \end{bmatrix} = \begin{bmatrix} s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{bmatrix} \Psi_{n+1}(t_0) \begin{bmatrix} \bar{s}_{n+1} \\ \vdots \\ \bar{s}_0 \end{bmatrix}. \tag{10}$$

We now formulate the main result:

Theorem 2.1. *Given $t_0 \in \mathbb{T}$ and $s_0, \dots, s_N$, let $n \leqslant (N+1)/2$ be the greatest integer such that the matrix $\mathbb{P}_n$ defined by formula* (9) *is positive semidefinite. In case $n \leqslant N/2$, let $p_{n+1,n}$ and $p_{n,n+1}$ be the numbers defined by the second formula in* (9) *and let B_n be as in* (10). *Then*

(1) *The problem $\mathbf{BP}_N$ has a unique solution if and only if $|s_0| = 1$, $\mathbb{P}_n$ is singular and, either*
 (a) $n = (N+1)/2$ *and* $\operatorname{rank} \mathbb{P}_n^{\mathbf{s}} = \operatorname{rank} \mathbb{P}_{n-1}^{\mathbf{s}}$, *or*
 (b) $n = N/2$, $p_{n+1,n} = \bar{p}_{n,n+1}$ *and* $\operatorname{rank} \mathbb{P}_n = \operatorname{rank}[\mathbb{P}_n \, B_n]$.
 In this case, the unique solution of the problem is a finite Blaschke product of degree equal $\operatorname{rank} \mathbb{P}_n$.
(2) *The problem $\mathbf{BP}_N$ has infinitely many solutions if and only if, either*
 (a) $|s_0| < 1$, *or*
 (b) $|s_0| = 1$, $\mathbb{P}_n > 0$ *and one of the following holds*:
  (i) $n = (N+1)/2$;
  (ii) $n = N/2$ *and* $t_0(p_{n+1,n} - \bar{p}_{n,n+1}) \geqslant 0$;
  (iii) $0 < n < N/2$ *and* $t_0(p_{n+1,n} - \bar{p}_{n,n+1}) > 0$.

(3) *Otherwise the problem has no solutions.*

Let us show that Theorem 1.4 follows from Theorem 2.1 upon letting $N = 2$ in the latter. Indeed, given $t_0 \in \mathbb{T}$ and $s_0, s_1, s_1, s_2 \in \mathbb{C}$ we can define the numbers,

$$\mathbb{P}_1 = p_{11} = \mathbb{H}_1 \Psi_1(t_0)\mathbb{U}_1^* = s_1 t_0 \bar{s}_0, \quad p_{21} = t_0 s_2 \bar{s}_0 \quad \text{and} \quad p_{12} = |s_1|^2 t_0 - s_1 \bar{s}_0 t_0^2 - s_2 \bar{s}_0 t_0^3, \tag{11}$$

by formulas (9). Since $N = 2$, the cases (1a), (2b(i)) and (2b(iii)) in Theorem 2.1 are not relevant. By part (1) in Theorem 2.1, the problem $\mathbf{BP}_2$ has a unique solution ($f \equiv s_0$) if and only if $|s_0| = 1$, $\mathbb{P}_1^{\mathbf{s}} = s_1 t_0 \bar{s}_0 = 0$ and $B_1 = p_{12} = 0$. It is not hard to see that these three conditions are equivalent to $|s_0| = 1$ and $s_1 = s_2 = 0$. By part (2) in Theorem 2.1, there are infinitely many functions $f \in \mathcal{S}$ satisfying (5) if and only if either $|s_0| < 1$ or $|s_0| = 1$, $\mathbb{P}_1 > 0$ and $t_0(p_{n+1,n} - \bar{p}_{n,n+1}) \geqslant 0$ and the latter conditions are equivalent to those in (6) which can be readily seen from (11).

We next observe that due to the upper triangular structure of the factors $\Psi_n(t_0)$ and $\mathbb{U}_n^*$ in (9), that $\mathbb{P}_k$ is the principal submatrix of $\mathbb{P}_n$ for every $k < n$. Part (1) in Theorem 2.1 can be formulated in the following more unified way:

Proposition 2.2. *The uniqueness occurs if and only if the matrix $\mathbb{P}_n$ of the maximally possible size (i.e., with $n = [\frac{N+1}{2}]$) is positive semidefinite (and singular) and admits a positive semidefinite extension $\mathbb{P}_{n+1}$ for an appropriate choice of s_{2n+1} (in case $N = 2n$) or of s_{2n+1} and s_{2n} (in case $N = 2n - 1$).*

Additional symmetry and rank conditions in part (1) of Theorem 2.1 guarantee that the above extension exists. As for part (2) of the theorem, we recall that the sufficient condition (2a) follows from Proposition 1.2 and the sufficient condition (2b(i)) has been established earlier (see e.g., [1,2,4,5,8]).

The algorithm determining whether or not there exists a Schur class function with prescribed boundary derivatives can be designed as follows. If $|s_0| \neq 1$, then the definitive answer comes up. If $|s_0| = 1$, it is not necessary to check positivity of all the matrices $\mathbb{P}_k$ for $k = 1, 2, \ldots$ to find the greatest integer n such that $\mathbb{P}_n \geqslant 0$. It suffices to get the greatest n such that $\mathbb{P}_n$ is Hermitian. If this Hermitian $\mathbb{P}_n$ is not positive semidefinite, then the problem $\mathbf{BP}_N$ has no solutions; this is a known fact which also can be deduced from Theorem 2.1. If $\mathbb{P}_n$ is positive semidefinite (singular), then we check one of the two possibilities indicated in part (1) depending on the parity of N. If $\mathbb{P}_n > 0$, then we verify exactly one of the three possibilities in part (2b). From the numerical point view this algorithm should be quite efficient since no matrix inversions are involved.

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