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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 315–320

Comptes Rendus Mathematique

Statistics

Signed symmetric covariation coefficient for alpha-stable dependence modeling

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Received 14 November 2008; accepted after revision 2 January 2009

Available online 14 February 2009

Presented by Paul Deheuvels

Abstract

We introduce a new measure of dependence between the components of a symmetric α-stable random vector that we call the signed symmetric covariation coefficient. We show that this coefficient satisfies the properties of the classical Pearson coefficient. Moreover, we show that in the case of sub-Gaussian random vectors, this coefficient coincide with the association parameter and the generalized association parameter. ***To cite this article: B. Garel, B. Kodia, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Coefficient de covariation symétrique signé pour une modélisation de la dépendance alpha-stable. On introduit ici une nouvelle mesure de dépendance entre les composantes d'un vecteur aléatoire α-stable symétrique appelé coefficient de covariation symétrique signé. On montre que ce coefficient satisfait les propriétés du coefficient de corrélation classique. De plus, on montre que dans le cas des vecteurs alétoires sous-gaussiens, ce coefficient coïncide avec le paramètre d'association et la version généralisée de ce paramètre appelée gap. ***Pour citer cet article : B. Garel, B. Kodia, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Trouver une mesure de dépendance appropriée est un problème récurrent lorsqu'on procède à une modélisation utilisant des lois stables non-gaussiennes. En effet, de telles distributions ne possèdent pas de moments d'ordre 2 et, comme conséquence immédiate, le concept de matrice de corrélation, qui permet de capter la structure de dépendence d'un vecteur aléatoire, n'a plus de sens. Il est donc nécessaire de définir d'autres coefficients de dépendance basés sur des moments plus petits que 2. Dans cette note, nous commençons par passer en revue quelques définitions de base et propriétés des vecteurs aléatoires α-stables. Nous présentons ensuite les mesures de dépendance utilisées jusqu'à présent pour définir la dépendance entre composantes d'un vecteur aléatoire α-stable symétrique. Nous introduisons un nouveau coefficient de dépendance que nous appelons coefficient de covariation symétrique signé. Ce coefficient, basé sur la covariation introduite par Miller [5], est donné par l'expression

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doi:10.1016/j.crma.2009.01.013

$$\mathrm{scov}(X_1, X_2) = \kappa_{(X_1,X_2)} \left| \frac{[X_1, X_2]_\alpha [X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha} \right|^{1/2},$$

où $[X_1, X_2]_\alpha$ définie par la relation (6) désigne la covariation de X_1 sur X_2, $\|\cdot\|_\alpha$ est la norme de la covariation, $\kappa_{(X_1,X_2)}$ est définie par (18), X_1 et X_2 étant des variables aléatoires symétriques α-stables avec $\alpha > 1$. De manière générale, ce coefficient a des propriétés similaires au coefficient de corrélation classique (Proposition 3.2). Dans la Proposition 4.3 on montre que dans le contexte des vecteurs aléatoires sous-gaussiens, ce coefficient coïncide avec le paramètre d'association introduit par Press [6] et le paramètre d'association généralisé proposé par Paulauskas [7].

1. Introduction

Many types of physical phenomena and financial data exhibit a very high variability and stable distributions are often used for their modeling. Since the seminal work of Mandelbrot (1960) who suggested the stable laws as possible models for the distributions of income and speculative prices, the interest in these laws greatly increased and now they are widely used in telecommunications and many other fields such as physics, biology, genetic and geology (Uchaikin and Zolotarev [9]).

Stable distributions are a rich class of probability distributions that include the Gaussian, Cauchy and Lévy distributions in a family that allows skewness and heavy tails. These laws, characterized by Paul Lévy (1924), are the only possible limiting laws for sums of independent, identically distributed random variables. While they present many attractive theoretical properties, a major problem in working with stable laws, both univariate and multivariate, is that except the three laws mentioned before, their densities cannot be written in a closed form. The only available information for a stable random vector is its characteristic function. Added to this drawback, the well-known problem that the stable non-Gaussian random vectors do not possess moments of second order limited their use. The concept of correlation matrix which allows to understand the association between the coordinates of a random vector, is meaningless here. Therefore we need other coefficients of dependence based on moments of order less than two.

Press [6] proposed an extended notion of correlation coefficient applicable to a family of symmetric multivariate stable laws so-called association parameter (a.p.). Then Paulauskas [7] proposed the generalized association parameter (gap). Kanter and Steiger [4] showed that, under some conditions, the conditional expectation of a stable variable given another one is linear. After that, Miller [5] and Cambanis and Miller [1] proposed a new dependence measure called covariation. The constant of linearity of conditional expectation has been expressed by means of this measure and then called the covariation coefficient. Garel et al. [3] introduced the symmetric covariation. In this Note we propose an other coefficient based on covariation that we call signed symmetric covariation coefficient and we show that in the case of sub-Gaussian random vectors, this new coefficient coincides with the association parameter and the gap.

This note is organized as follow: Section 2 is devoted to a reminder of basic definitions and some properties of stable random vectors, the dependence measures and the above mentioned coefficients. In Section 3, we introduce the signed symmetric covariation coefficient and we give its first properties. Other properties of this new coefficient are discussed in the context of sub-Gaussian random vectors in Section 4.

2. Review of bivariate alpha-stable random vectors and coefficients of dependence

For our purposes it is convenient to define stable random variables and vectors by means of their characteristic function.

Definition 2.1. A random variable X is said to have a stable distribution if there are parameters $0 < \alpha \leqslant 2$, $\gamma \geqslant 0$, $-1 \leqslant \beta \leqslant 1$, and δ real such that its characteristic function has the following form:

$$E \exp \mathrm{i}\theta X = \exp\left\{-\gamma^\alpha |\theta|^\alpha \left[1 + \mathrm{i}\beta \,\mathrm{sign}(\theta) w(\theta, \alpha)\right] + \mathrm{i}\delta\theta\right\}, \tag{1}$$

where

$$w(\theta, \alpha) = \begin{cases} -\tan \dfrac{\pi\alpha}{2} & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi} \ln |\theta| & \text{if } \alpha = 1, \end{cases}$$

and

$$\operatorname{sign}(\theta)=\begin{cases}1 & \text{if } \theta>0,\\ 0 & \text{if } \theta=0,\\ -1 & \text{if } \theta<0.\end{cases}$$

The parameter α is called the characteristic exponent or the index of stability, β is a measure of skewness, γ is a scale parameter, and δ is a location parameter. We will use the notation $X \sim S_\alpha(\gamma,\beta,\delta)$. When $\beta=\delta=0$, X is said symmetric α-stable ($S\alpha S$, which means that X and $-X$ have the same distribution) and its characteristic function takes the particular simple form:

$$E\exp \mathrm{i}\theta X=\exp\{-\gamma^\alpha|\theta|^\alpha\}. \tag{2}$$

The following theorem extend to $\mathbb{R}^2$ the definition of a stable random variable:

Theorem 2.2. *Let* $0<\alpha<2$. *Then* $\mathbf{X}=(X_1,X_2)$ *is an* α*-stable random vector in* $\mathbb{R}^2$ *if and only if there exists a finite measure* $\boldsymbol{\Gamma}$ *on the unit circle* $S_2=\{s\in\mathbb{R}^2\colon\ \|\mathbf{s}\|=1\}$ *and a vector* $\boldsymbol{\delta}$ *such that for all* $\boldsymbol{\theta}\in\mathbb{R}^2$

$$E\exp\bigl(\mathrm{i}(\boldsymbol{\theta},\mathbf{X})\bigr)=\exp\left\{-\int_{S_2}\bigl|(\boldsymbol{\theta},\mathbf{s})\bigr|^\alpha\bigl[1+\mathrm{i}\operatorname{sign}(\boldsymbol{\theta},\mathbf{s})w\bigl((\boldsymbol{\theta},s),\alpha\bigr)\bigr]\boldsymbol{\Gamma}(\mathrm{d}\mathbf{s})+\mathrm{i}(\boldsymbol{\theta},\boldsymbol{\delta})\right\}. \tag{3}$$

Here $(\boldsymbol{\theta},\mathbf{s})$ denotes the canonical inner product of $\mathbb{R}^2$. The measure $\boldsymbol{\Gamma}$ is called spectral measure of the α-stable random vector $\mathbf{X}$ and the pair $(\boldsymbol{\Gamma},\boldsymbol{\delta})$ is unique. The vector $\mathbf{X}$ is symmetric if and only if $\boldsymbol{\delta}=0$ and $\boldsymbol{\Gamma}$ is a symmetric measure on S_2. In this case, its characteristic function is given by

$$E\exp\bigl\{\mathrm{i}(\boldsymbol{\theta},\mathbf{X})\bigr\}=\exp\left\{-\int_{S_2}\bigl|(\boldsymbol{\theta},\mathbf{s})\bigr|^\alpha\boldsymbol{\Gamma}(\mathrm{d}\mathbf{s})\right\}. \tag{4}$$

For any vector $\mathbf{u}\in\mathbb{R}^2$ the projection $(\mathbf{u},\mathbf{X})=\sum_{k=1}^2 u_kX_k$ has an univariate α-stable distribution $S_\alpha(\gamma_\mathbf{u},\beta_\mathbf{u},\delta_\mathbf{u})$. The spectral measure determines the projection parameter function $\gamma(\mathbf{u})$ by:

$$\gamma^\alpha(\mathbf{u})=\gamma^\alpha(u_1,u_2)=\int_{S_2}\bigl|(\mathbf{u},\mathbf{s})\bigr|^\alpha\boldsymbol{\Gamma}(\mathrm{d}s). \tag{5}$$

Unless otherwise indicated, for the remainder of the note we are working with symmetric α-stable random vectors with $\alpha>1$.

The covariation, introduced by Miller (1978), is defined as follows:

Definition 2.3. Let X_1 and X_2 be jointly $S\alpha S$ with $\alpha>1$ and let $\boldsymbol{\Gamma}$ be the spectral measure of the random vector $\mathbf{X}=(X_1,X_2)$. The covariation of X_1 on X_2 is the real number

$$[X_1,X_2]_\alpha=\int_{S_2}s_1s_2^{\langle\alpha-1\rangle}\boldsymbol{\Gamma}(\mathrm{d}\mathbf{s}), \tag{6}$$

with $a^{\langle p\rangle}=|a|^p\operatorname{sign}(a)$ is called signed power.

This definition is equivalent to:

$$[X_1,X_2]_\alpha=\frac{1}{\alpha}\frac{\partial\gamma^\alpha(\theta_1,\theta_2)}{\partial\theta_1}\bigg|_{\theta_1=0,\,\theta_2=1}, \tag{7}$$

where θ_1 and θ_2 are real numbers and $\gamma(\theta_1,\theta_2)$ is the scale parameter of the random variable $Y=\theta_1X_1+\theta_2X_2$. It is well known that although the covariation is linear in its first argument, it is, in general, not linear in its second argument and not symmetric in its arguments. We also have

$$[X_1,X_1]_\alpha=\int_{S_2}|s_1|^\alpha\boldsymbol{\Gamma}(\mathrm{d}\mathbf{s})=\gamma_{X_1}^\alpha. \tag{8}$$

We denote

$$\|X_1\|_\alpha = \big([X_1, X_1]_\alpha\big)^{1/\alpha} = \gamma_{X_1}, \tag{9}$$

where γ_{X_1} is the scale parameter of the $S\alpha S$ random variable X_1. Then $\|\cdot\|_\alpha$ defines a norm called covariation norm. When X_1 and X_2 are independent, $[X_1, X_2]_\alpha = 0$. Proofs of these properties and other details are given in Samorodnitsky and Taqqu [8, pp. 87–97].

The following lemma establishes an important result which shows how the covariation $[X_1, X_2]_\alpha$ is related to the joint moment $EX_1X_2^{\langle p-1\rangle}$:

Lemma 2.4. *Let (X_1, X_2) be $S\alpha S$ with $\alpha > 1$. Then for all $1 \leqslant p < \alpha$,*

$$\frac{EX_1X_2^{\langle p-1\rangle}}{E|X_2|^p} = \frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha}. \tag{10}$$

Proof. The demonstration is detailed in d'Estampes [2, pp. 35–37]. □

The covariation coefficient of X_1 on X_2 is the quantity:

$$\lambda_{X_1,X_2} = \frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha}. \tag{11}$$

It is the coefficient of the linear regression $E(X_1|X_2)$. This coefficient is not symmetric and may be unbounded. The symmetric coefficient of covariation between X_1 and X_2 is given by:

$$\mathrm{Corr}_\alpha(X_1, X_2) = \lambda_{X_1,X_2}\lambda_{X_2,X_1} = \frac{[X_1, X_2]_\alpha[X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha\|X_2\|_\alpha^\alpha}. \tag{12}$$

This coefficient is symmetric, bounded and vanishes when X_1 and X_2 are independent (Garel et al. [3]).

Press [6] proposed a measure of association between coordinates of a bivariate symmetric stable vector. For a bivariate stable vector with characteristic function

$$\begin{aligned} E\exp\big(\mathrm{i}(\boldsymbol{\theta}, \mathbf{X})\big) &= \exp\left\{-\sum_{i=1}^{m}(\boldsymbol{\theta}\boldsymbol{\Omega}_i\boldsymbol{\theta}')^{\alpha/2}\right\} \\ &= \exp\left\{-\sum_{i=1}^{m}\big(w_{11}(i)\theta_1^2 + 2w_{12}(i)\theta_1\theta_2 + w_{22}(i)\theta_2^2\big)^{\alpha/2}\right\}, \end{aligned} \tag{13}$$

with $\boldsymbol{\Omega}_i$, $i = 1, \ldots, m$, a 2×2 symmetric positive semidefinite matrix, the association parameter (a.p.) ρ is defined as follows:

$$\rho = \frac{\sum_{i=1}^m w_{12}(i)}{[(\sum_{i=1}^m w_{11}(i))(\sum_{i=1}^m w_{22}(i))]^{1/2}}. \tag{14}$$

It is shown that for $\alpha = 2$ the a.p. coincides with the ordinary correlation coefficient of the bivariate Gaussian random vector and for $0 < \alpha < 2$ the a.p. possesses all the properties of a correlation coefficient. But since the expression (13) does not represent all symmetric stable bivariate distributions, Paulauskas [7] introduced another concept, which on the one hand, would be applicable to all symmetric stable random vector in $\mathbb{R}^2$ and on the other hand would have all the properties as ρ.

Let (X_1, X_2) be $S\alpha S$, $0 < \alpha \leqslant 2$ and $\boldsymbol{\Gamma}$ its spectral measure on the unit circle S_2. Let (U_1, U_2) be a random vector on S_2 with probability distribution $\tilde{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}/\boldsymbol{\Gamma}(S_2)$. Because of the symmetry of $\boldsymbol{\Gamma}$, one has $EU_1 = EU_2 = 0$. The generalized association parameter (gap) was defined by Paulauskas as:

$$\tilde{\rho} = \frac{EU_1U_2}{(EU_1^2EU_2^2)^{1/2}}. \tag{15}$$

It is a measure of dependence for (X_1, X_2). For a bivariate stable vector with characteristic function (4) the gap $\tilde{\rho}$ has the following properties valid for all $0 < \alpha \leqslant 2$:

(i) always $-1 \leqslant \tilde{\rho} \leqslant 1$ and if a distribution corresponds to a random vector with independent coordinates, then $\tilde{\rho} = 0$;
(ii) $|\tilde{\rho}| = 1$ if and only if the distribution is concentred on a line;
(iii) for $\alpha = 2$, $\tilde{\rho}$ coincides with the correlation coefficient of the Gaussian random vector;
(iv) $\tilde{\rho}$ is independent of α and depends only on the spectral measure $\mathbf{\Gamma}$;
(v) if the characteristic function of (X_1, X_2) is given by

$$\varphi(\boldsymbol{t}) = \exp\{-C(\gamma_1^2 t_1^2 + 2r\gamma_1\gamma_2 t_1 t_2 + \gamma_2^2 t_2^2)^{\alpha/2}\}, \tag{16}$$

where C is an appropriate constant, then r is the gap.

3. Signed symmetric covariation coefficient and its properties

We now introduce our new measure of dependence.

Definition 3.1. Let (X_1, X_2) be a bivariate $S\alpha S$ random vector with $\alpha > 1$. The signed symmetric covariation coefficient between X_1 and X_2 is the quantity:

$$\operatorname{scov}(X_1, X_2) = \kappa_{(X_1,X_2)} \left| \frac{[X_1, X_2]_\alpha [X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha} \right|^{1/2}, \tag{17}$$

where

$$\kappa_{(X_1,X_2)} = \begin{cases} \operatorname{sign}([X_1, X_2]_\alpha) & \text{if } \left|\frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha}\right| \geqslant \left|\frac{[X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha}\right|, \\ \operatorname{sign}([X_2, X_1]_\alpha) & \text{if } \left|\frac{[X_1, X_2]_\alpha}{\|X_2\|_\alpha^\alpha}\right| < \left|\frac{[X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha}\right|. \end{cases} \tag{18}$$

So $\kappa_{(X_1,X_2)}$ denotes the sign of the coefficient of covariation which has the greatest absolute value. Using equality (10) with $p = 1$, we see that this coefficient can be expressed by:

$$\operatorname{scov}(X_1, X_2) = \kappa_{(X_1,X_2)} \left| \frac{(EX_1 \operatorname{sign}(X_2))(EX_2 \operatorname{sign}(X_1))}{E|X_1|E|X_2|} \right|^{1/2}. \tag{19}$$

This last expression will give us a way to estimate the signed symmetric covariation coefficient without knowing the value of α.

Proposition 3.2. *Let (X_1, X_2) be a bivariate $S\alpha S$ random vector with $\alpha > 1$. The signed symmetric covariation coefficient has the following properties*:

(i) $-1 \leqslant \operatorname{scov}(X_1, X_2) \leqslant 1$ *and if X_1, X_2 are independent, then* $\operatorname{scov}(X_1, X_2) = 0$;
(ii) *for all $a \neq 0$,* $|\operatorname{scov}(X, aX)| = 1$;
(iii) *let a and b two non-zero reals, X_1 and X_2 such that* $[X_1, X_2]_\alpha \neq 0$ *and* $[X_2, X_1]_\alpha \neq 0$, *then* $\operatorname{scov}(aX_1, bX_2) = \pm\operatorname{scov}_\alpha(X_1, X_2)$;
(iv) *for $\alpha = 2$,* $\operatorname{scov}(X_1, X_2)$ *coincides with the usual correlation coefficient.*

Note that if $|\frac{[aX_1,bX_2]_\alpha}{\|bX_2\|_\alpha^\alpha}| \geqslant |\frac{[bX_2,aX_1]_\alpha}{\|aX_1\|_\alpha^\alpha}|$ and $|a| \leqslant |b|$ then $\kappa_{(aX_1,bX_2)} = \operatorname{sign}(ab)\operatorname{sign}([X_1, X_2]_\alpha)$ and $\kappa_{(X_1,X_2)} = \operatorname{sign}([X_1, X_2]_\alpha)$. It implies that

$$\operatorname{scov}(aX_1, bX_2) \begin{cases} \operatorname{scov}(X_1, X_2) & \text{if } a \text{ and } b \text{ have the same sign,} \\ -\operatorname{scov}(X_1, X_2) & \text{if not.} \end{cases}$$

The other cases can be obtained in a similar way.

4. Signed symmetric covariation coefficient in sub-Gaussian case

We begin this section by the definition of a sub-Gaussian random vector.

Definition 4.1. Let $0 < \alpha < 2$, let G_1, G_2 be zero mean jointly normal random variables and let A be a positive random variable such that $A \sim S_{\alpha/2}((\cos \frac{\pi\alpha}{4})^{2/\alpha}, 1, 0)$, independent of (G_1, G_2), then $\mathbf{X} = A^{1/2}\mathbf{G} = (A^{1/2}G_1, A^{1/2}G_2)$ is a sub-Gaussian random vector with underlying Gaussian vector $\mathbf{G} = (G_1, G_2)$.

The characteristic function of $\mathbf{X}$ is given by:

$$E \exp\left\{ \mathrm{i} \sum_{k=1}^{2} \theta_k X_k \right\} = \exp\left\{ -\left| \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \theta_i \theta_j R_{ij} \right|^{\alpha/2} \right\}, \tag{20}$$

where $R_{ij} = EG_iG_j$, $i, j = 1, 2$, are the covariances of the underlying Gaussian random vector $\mathbf{G}$ (Samorodnitsky and Taqqu, [8, p. 78]).

Before proving that, in this context, the gap of Paulauskas, the a.p. of Press and the signed symmetric covariation coefficient coincide, we first establish the following lemma:

Lemma 4.2. *Let* $0 < \alpha < 2$ *and* $\mathbf{X}$ *as in Definition* 4.1. *Then the gap of and the a.p. between the components of* $\mathbf{X}$ *coincide with the correlation coefficient between the components of* $\mathbf{G}$.

Proposition 4.3. *Let* $1 < \alpha < 2$ *and* $\mathbf{X}$ *as in Definition* 4.1. *Then the signed symmetric covariation coefficient between the components of* $\mathbf{X}$ *coincides with the gap and the a.p. between the components of the same vector.*

Property 4.4. *Let* $1 < \alpha < 2$ *and* $\mathbf{X}$ *as in Definition* 4.1. *If* $|\mathrm{scov}(X_1, X_2)| = 1$ *then the distribution of* $\mathbf{X}$ *is concentrated on a line.*

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