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Probability Theory

An estimate for the vorticity of the Navier–Stokes equation

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Abstract

Let $\vec{u}(\cdot,t)$ be a strong solution of the Navier–Stokes equation on 3-dimensional torus $\mathbb{T}^3$, and $\vec{\omega}(\cdot,t)=\nabla\times\vec{u}(\cdot,t)$ be the vorticity. In this Note we show that

$$\left\|\vec{\omega}(\cdot,t)\right\|_1+\frac{\sqrt{2}}{4\nu}\left\|\vec{u}(\cdot,t)\right\|_2^2$$

is decreasing in t as long as the solution $\vec{u}(\cdot,t)$ exists, where $\nu>0$ is the viscosity constant and $\|\cdot\|_q$ denotes the L^q-norm. ***To cite this article: Z. Qian, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une estimation de la vorticité de l'équation de Navier–Stokes. Supposons que $\vec{u}(\cdot,t)$ soit une solution de l'équation de Navier–Stokes sur le torus $\mathbb{T}^3$ de la dimension 3, et soit $\vec{\omega}(\cdot,t)=\nabla\times\vec{u}(\cdot,t)$ la vorticité, nous démontrons dans cette Note que l'application

$$\left\|\vec{\omega}(\cdot,t)\right\|_1+\frac{\sqrt{2}}{4\nu}\left\|\vec{u}(\cdot,t)\right\|_2^2$$

est une fonction décroissante en t. ***Pour citer cet article : Z. Qian, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. The main result

Consider the Navier–Stokes equation

$$\frac{\partial}{\partial t}\vec{u}+(\vec{u}\cdot\nabla)\vec{u}=\nu\Delta\vec{u}-\nabla p,\quad \nabla\cdot\vec{u}=0 \tag{1}$$

on 3-dimensional torus $\mathbb{T}^3$, where $\nu>0$ is the viscosity constant. For each $t\geqslant 0$, $\vec{u}(\cdot,t)$ is a vector field on $\mathbb{T}^3$ and $p(\cdot,t)$ is a scalar function on $\mathbb{T}^3$ called the pressure at instance t.

Given a smooth initial velocity $\vec{u}(\cdot,0)$, a unique smooth solution of (1) exists up to some time T^* dependent on the initial data, see for example [5,6,8,10] for a proof. Few *a priori* estimates for strong solutions of Navier–Stokes

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equations on spaces of dimension 3 exist in the literature, and the global existence (in time) of strong solutions remains open. On the other hand, Leray [7] and Hopf [4] demonstrated the global existence of weak solutions which in addition obey an energy inequality (see (4) below), but uniqueness of weak solutions remains open. It is known however that any Leray–Hopf's weak solution coincides with the strong solution on time interval $[0, T^*)$, and is indeed smooth as long as $t < T^*$.

Let $\vec{\omega} = \nabla \times \vec{u}$ be the vorticity. It is easy to derive from the Navier–Stokes equation that

$$\frac{\partial}{\partial t}|\vec{u}|^2 = 2\nu\vec{u}\cdot\Delta\vec{u} - \vec{u}\cdot\nabla|\vec{u}|^2 - 2\vec{u}\cdot\nabla p. \tag{2}$$

Integrating above equation over $\mathbb{T}^3$ and performing integration by parts, one obtains an energy balance identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}\|_{L^2}^2 = -2\nu\int_{\mathbb{T}^3}|\nabla\vec{u}|^2 = -2\nu\int_{\mathbb{T}^3}|\vec{\omega}|^2. \tag{3}$$

Integrating (3) over $[0, t]$ one obtains the energy inequality

$$\big\|\vec{u}(\cdot,t)\big\|_{L^2}^2 + 2\nu\int_0^t\int_{\mathbb{T}^3}\big|\vec{\omega}(\cdot,s)\big|^2 \leqslant \big\|\vec{u}(\cdot,0)\big\|_{L^2}^2 \tag{4}$$

which takes exactly the same form for solutions of the heat equation. The energy inequality (4) has been known for quite long time, and has been the important tool in the study of Navier–Stokes equations.

In this Note, we demonstrate that the energy balance equation (3) indeed contains more information than decoded in the energy inequality (4), and deduce a monotonicity (in time variable t) of the L^1-norm of the vorticity $\vec{\omega}$. Let us mention that *a priori* estimates (point-wise in time t) of the L^2-norm $\|\vec{\omega}(\cdot,t)\|_2$ would yield the global existence of a strong solution to the Navier–Stokes equation. Indeed a point-wise (in time) estimate for $\|\vec{\omega}(\cdot,t)\|_{3/2}$ would be enough to settle the global existence. For further details, see, for example, [2,3,9], and in particular [1] for a local L^1-estimate for the vorticity, and the literature therein.

An L^2-estimate for the vorticity $\vec{\omega}$ is still missing, we are nevertheless able to establish an estimate (point-wise in t) for the L^1-norm of the vorticity. The following is our main result:

Theorem 1. *Let $\vec{u}(\cdot,t)$ be the strong solution of the Navier–Stokes equation on $\mathbb{T}^3$ up to time T^*. Then*

$$t \to \big\|\vec{\omega}(\cdot,t)\big\|_{L^1} + \frac{\sqrt{2}}{4\nu}\big\|\vec{u}(\cdot,t)\big\|_{L^2}^2$$

is decreasing on $[0, T^*)$.

The next section is devoted to the proof of Theorem 1.

2. Proof of the main result

Taking curl operation on both sides of the Navier–Stokes equation (1), one obtains the vorticity equation

$$\frac{\partial}{\partial t}\vec{\omega} + (\vec{u}\cdot\nabla)\vec{\omega} = \nu\Delta\vec{\omega} + S(\vec{\omega}) \tag{5}$$

where $S(\vec{\omega}) = S_j^i\omega^j$, $S_j^i = \frac{1}{2}(\nabla_j u^i + \nabla_i u^j)$ is the symmetric tensor of the rate-of-strain. The pressure p does not appear explicitly in (5). It follows that the *enstropy*, $|\vec{\omega}|^2$, evolves according to the non-linear partial differential equation

$$\frac{\partial}{\partial t}|\vec{\omega}|^2 + (\vec{u}\cdot\nabla)|\vec{\omega}|^2 = \nu\Delta|\vec{\omega}|^2 + 2\vec{\omega}\cdot S(\vec{\omega}) - 2\nu|\nabla\vec{\omega}|^2. \tag{6}$$

Similarly, taking the divergence both sides of the Navier–Stokes equation, we see that $p(\cdot,t)$ at each instance t is a solution to the Poisson equation $\Delta p = -\nabla\cdot(\vec{u}\cdot\nabla\vec{u})$. In terms of the vorticity $\vec{\omega}$ and the symmetric tensor $S = (S_j^i)$, the Poisson equation may be written as

$$\Delta p = \frac{1}{2}|\vec{\omega}|^2 - |S|^2, \tag{7}$$

where $|S|$ is the Hilbert–Schmidt norm of (S^i_j). Integrating (7) over $\mathbb{T}^3$ one obtains that

$$\|\vec{\omega}\|_{L^2} = \sqrt{2}\|S\|_{L^2}. \tag{8}$$

Lemma 2. *Let $q \geqslant 1$ be a constant. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\omega}\|^q_{L^q} \leqslant -\frac{4(q-1)}{q}\nu \int\limits_{\mathbb{T}^3} \big|\nabla|\vec{\omega}|^{q/2}\big|^2 + q\int\limits_{\mathbb{T}^3} |\vec{\omega}|^q |S| \tag{9}$$

as long as $t < T^*$.

Proof. We observe that

$$\begin{aligned}
\big|\nabla|\vec{\omega}|^2\big|^2 &= \sum_i \Big(\nabla_i\Big(\sum_j \omega_j^2\Big)\Big)^2 \\
&= 4\sum_i\Big(\sum_j \omega_j \nabla_i \omega_j\Big)^2 \\
&\leqslant 4|\vec{\omega}|^2|\nabla\vec{\omega}|^2 \\
&\leqslant 4\big(|\vec{\omega}|^2+\varepsilon\big)|\nabla\vec{\omega}|^2
\end{aligned}$$

for every $\varepsilon > 0$, so that

$$|\nabla\vec{\omega}|^2 \geqslant \frac{|\nabla|\vec{\omega}|^2|^2}{4(|\vec{\omega}|^2+\varepsilon)}. \tag{10}$$

Let $L = \nu\Delta - \vec{u}\cdot\nabla$ and Ψ be a differentiable function on $[0,\infty)$. Then

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - L\Big)\Psi\big(|\vec{\omega}|^2\big) &= \Psi'\Big(\frac{\partial}{\partial t} - L\Big)|\vec{\omega}|^2 - \nu\Psi''\big|\nabla|\vec{\omega}|^2\big|^2 \\
&= -2\nu\Psi'|\nabla\vec{\omega}|^2 - \nu\Psi''\big|\nabla|\vec{\omega}|^2\big|^2 + 2\Psi'\vec{\omega}\cdot S(\vec{\omega}).
\end{aligned} \tag{11}$$

Choose $\Psi_\varepsilon(x) = (x+\varepsilon)^{q/2}$ where $q \geqslant 1$ and $\varepsilon > 0$, so that

$$\Psi'_\varepsilon(x) = \frac{q}{2}(x+\varepsilon)^{q/2-1} > 0$$

and

$$\Psi''_\varepsilon(x) = \frac{p}{2}\Big(\frac{q}{2}-1\Big)(x+\varepsilon)^{q/2-2},$$

and set $F_\varepsilon = (|\vec{\omega}|^2+\varepsilon)^{q/2}$ for simplicity. Then $F_\varepsilon = \Psi_\varepsilon(|\vec{\omega}|^2)$ and

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - L\Big)F_\varepsilon &= -2\nu\Psi'_\varepsilon\Big\{|\nabla\vec{\omega}|^2 + \frac{1}{2}\Big(\frac{q}{2}-1\Big)\frac{|\nabla|\vec{\omega}|^2|^2}{|\vec{\omega}|^2+\varepsilon}\Big\} + 2\Psi'_\varepsilon\vec{\omega}\cdot S(\vec{\omega}) \\
&\leqslant -2\nu\Psi'_\varepsilon\Big\{\frac{1}{4}\frac{|\nabla|\vec{\omega}|^2|^2}{|\vec{\omega}|^2+\varepsilon} + \frac{1}{2}\Big(\frac{q}{2}-1\Big)\frac{|\nabla|\vec{\omega}|^2|^2}{|\vec{\omega}|^2+\varepsilon}\Big\} + 2\Psi'_\varepsilon\vec{\omega}\cdot S(\vec{\omega}) \\
&= -2\nu\Psi'_\varepsilon\frac{q-1}{4}\frac{|\nabla|\vec{\omega}|^2|^2}{|\vec{\omega}|^2+\varepsilon} + 2\Psi'_\varepsilon\vec{\omega}\cdot S(\vec{\omega})
\end{aligned} \tag{12}$$

where the inequality follows from (10). Integrating above inequality over $\mathbb{T}^3$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{T}^3} F_\varepsilon \leqslant -\frac{q-1}{2}\nu\int\limits_{\mathbb{T}^3} \Psi'_\varepsilon\frac{|\nabla|\vec{\omega}|^2|^2}{|\vec{\omega}|^2+\varepsilon} + 2\int\limits_{\mathbb{T}^3} \Psi'_\varepsilon\vec{\omega}\cdot S(\vec{\omega}). \tag{13}$$

Letting $\varepsilon \downarrow 0$ we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}|\vec{\omega}|^q \leqslant -\frac{q-1}{2}\nu\int_{\mathbb{T}^3}\Psi'\frac{|\nabla|\vec{\omega}|^2|^2}{|\vec{\omega}|^2}+2\int_{\mathbb{T}^3}\Psi'\vec{\omega}\cdot S(\vec{\omega}) \tag{14}$$

which is equivalent to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\omega}\|_{L^q}^q \leqslant -4\left(1-\frac{1}{q}\right)\nu\int_{\mathbb{T}^3}\left|\nabla|\vec{\omega}|^{q/2}\right|^2+q\int_{\mathbb{T}^3}|\vec{\omega}|^{q-2}\vec{\omega}\cdot S(\vec{\omega}). \qquad \square \tag{15}$$

We are now in a position to establish the L^1-estimate for the vorticity. Letting $q=1$ in (9), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\omega}\|_{L^1} &\leqslant \int_{\mathbb{T}^3}|\vec{\omega}\|S| \leqslant \|\vec{\omega}\|_{L^2}\|S\|_{L^2}\\ &= \frac{1}{\sqrt{2}}\|\vec{\omega}\|_{L^2}^2\end{aligned} \tag{16}$$

which does not depend on the viscosity ν, where the second inequality follows from the Cauchy–Schwarz inequality, and the equality comes from (8). Next, we use the energy identity (3) and replace $\|\vec{\omega}\|_{L^2}^2$ by $-\frac{1}{2\nu}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}\|_{L^2}^2$ in (16). We then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{\omega}\|_{L^1} \leqslant -\frac{\sqrt{2}}{4\nu}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{u}\|_{L^2}^2$$

which proves Theorem 1.

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