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Mathematical Analysis

Quasicrystals are sets of stable sampling

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Abstract

Irregular sampling and "stable sampling" of band-limited functions have been studied by H.J. Landau [H.J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. 117 (1967) 37–52]. We prove that quasicrystals are sets of stable sampling. ***To cite this article: B. Matei, Y. Meyer, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Les quasicristaux sont des ensembles d'échantillonnage stables. Répondant à une question posée par H.J. Landau [H.J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. 117 (1967) 37–52] sur le problème de l'échantillonnage irrégulier des fonctions "band-limited", nous prouvons que les quasicristaux sont des ensembles d'échantillonnage stable. ***Pour citer cet article : B. Matei, Y. Meyer, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Soit $K \subset \mathbb{R}^n$ un ensemble compact et soit $E_K \subset L^2(\mathbb{R}^n)$ le sous-espace de $L^2(\mathbb{R}^n)$ composé de toutes les fonctions $f \in L^2(\mathbb{R}^n)$ dont la transformée de Fourier $\hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\,dx$ est nulle hors de K. En utilisant la terminologie introduite dans [2], [5] un ensemble $\Lambda \subset \mathbb{R}^n$ est un "ensemble d'échantillonnage stable" pour E_K s'il existe une constante C telle que pour toute $f \in E_K$ on ait

$$\|f\|_2^2 \leqslant C \sum_{\lambda\in\Lambda} \left|f(\lambda)\right|^2. \tag{1}$$

Si $n = 1$ et si K est un intervalle de $\mathbb{R}$, le problème a été étudié et complètement résolu dans [1].

Pour $n \geqslant 1$ et K arbitraire, H.J. Landau [2] a démontré que (1) implique $\underline{\text{dens}}\,\Lambda \geqslant |K|$. Mais la réciproque n'est pas vraie et $|K| < \underline{\text{dens}}\,\Lambda$ n'implique pas (1) même dans le cas le plus simple où $\Lambda = \mathbb{Z}^n$. Nous allons prouver le résultat suivant : *Pour tout quasicrystal simple $\Lambda \subset \mathbb{R}^n$ et tout ensemble compact $K \subset \mathbb{R}^n$, la condition $|K| <$ dens Λ entraîne* (1).

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1. Introduction

Let $K \subset \mathbb{R}^n$ be a compact set and $E_K \subset L^2(\mathbb{R}^n)$ be the translation invariant subspace of $L^2(\mathbb{R}^n)$ consisting of all $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\,dx$ is supported by K. We now follow [2].

Definition 1.1. A set $\Lambda \subset \mathbb{R}^n$ has the property of stable sampling for E_K if there exists a constant C such that

$$f \in E_K \Rightarrow \|f\|_2^2 \leqslant C \sum_{\lambda\in\Lambda} \left|f(\lambda)\right|^2. \tag{2}$$

In other words any "band-limited" $f \in E_K$ can be reconstructed from its sampling $f(\lambda)$, $\lambda \in \Lambda$. Here is an equivalent definition. Let $L^2(K)$ be the space of all restrictions to K of functions in $L^2(\mathbb{R}^n)$. Then $\Lambda \subset \mathbb{R}^n$ is a set of stable sampling for E_K if and only if the collection of functions $\exp(2\pi i\lambda\cdot x)$, $\lambda \in \Lambda$, is a frame of $L^2(K)$.

Definition 1.2. A set Λ has the property of stable interpolation for E_K if there exists a constant C such that

$$\sum_{\lambda\in\Lambda} \left|c(\lambda)\right|^2 \leqslant C\|f\|^2_{L^2(K)} \tag{3}$$

for every finite trigonometric sum $f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\exp(2\pi i\lambda\cdot x)$.

In the one dimensional case and when K is an interval, A. Beurling (see [1]) proved the following:

Proposition 1.3. *Let Λ be an increasing sequence λ_j, $j \in \mathbb{Z}$, of real numbers such that $\lambda_{j+1} - \lambda_j \geqslant \beta > 0$. Let $\overline{\mathrm{dens}}\,\Lambda = \lim_{R\to\infty} R^{-1} \sup_{x\in\mathbb{R}} \mathrm{card}\{\Lambda \cap [x, x+R]\}$ be the upper density of Λ. The lower density is defined by replacing upper bounds by lower bounds. Then for any interval K, $|K| < \underline{\mathrm{dens}}\,\Lambda$ implies* (2) *and $|K| > \overline{\mathrm{dens}}\,\Lambda$ implies* (3).

Returning to the general case $K \subset \mathbb{R}^n$ H.J. Landau proved in [2] that (2) implies $\underline{\mathrm{dens}}\,\Lambda \geqslant |K|$ and (3) implies $\overline{\mathrm{dens}}\,\Lambda \leqslant |K|$. These necessary conditions are not sufficient. Indeed $|K| < \underline{\mathrm{dens}}\,\Lambda$ does not even imply (2) when $\Lambda = \mathbb{Z}^n$. The following result shows that Landau's necessary conditions are sufficient for some sets Λ.

Theorem 1.4. *Let $\Lambda \subset \mathbb{R}^n$ be a simple quasicrystal and $K \subset \mathbb{R}^n$ be a compact set. Then $|K| < \mathrm{dens}\,\Lambda$ implies* (2)*. If K is Riemann integrable, then $|K| > \mathrm{dens}\,\Lambda$ implies* (3).

A compact $K \subset \mathbb{R}^n$ is a Riemann integrable if the Lebesgue measure of its boundary is 0.

We now define a simple quasicrystal as in [2] or [3]. Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ be a lattice and if $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, let us write $p_1(x,t) = x$, $p_2(x,t) = t$. We now assume that p_1 once restricted to Γ is an injective mapping onto $p_1(\Gamma) = \Gamma_1$. We make the same assumption on p_2. We furthermore assume that $p_1(\Gamma)$ is dense in $\mathbb{R}^n$ and $p_2(\Gamma)$ is dense in $\mathbb{R}$. The dual lattice of Γ is denoted Γ^* and is defined by $x\cdot y \in \mathbb{Z}$, $x \in \Gamma$, $y \in \Gamma^*$. We use the following notations. For $\gamma = (x,t) \in \Gamma$ we write $t = \tilde{x}$, $\tilde{t} = x$. The same notations are used for the two components of $\gamma^* \in \Gamma^*$. If $I = [-\alpha, \alpha]$, the simple quasicrystal $\Lambda_I \subset \mathbb{R}^n$ is defined by

$$\Lambda_I = \left\{p_1(\gamma);\ \gamma \in \Gamma,\ p_2(\gamma) \in I\right\}. \tag{4}$$

2. Proof of Theorem 1.4

If $K \subset \mathbb{R}^n$ is a compact set, $M_K \subset \mathbb{R}$ is defined by

$$M_K = \left\{p_2(\gamma^*);\ \gamma^* \in \Gamma^*,\ p_1(\gamma^*) \in K\right\}. \tag{5}$$

The density of Λ_I is uniform and is given by $c|I|$ where $c = c(\Gamma)$ and similarly the density of M_K is $|K|/c$ when K is Riemann integrable [3,4]. Therefore $|K| < \mathrm{dens}\,\Lambda_I$ implies $|I| > \mathrm{dens}\,M_K$ which will be crucial in what follows. We sort the elements of M_K in increasing order and denote the corresponding sequence by $\{m_k;\ k \in \mathbb{Z}\}$. Then we have [3,4]:

Lemma 2.1. *The sequence $\{\tilde{m}_k;\ k \in \mathbb{Z}\}$ is equidistributed on K.*

We now prove our main result.

We replace K by a larger compact set still denoted by K which is Riemann integrable and still satisfies $|K| < \operatorname{dens} \Lambda$. By a standard density argument we can assume $\hat{f} \in \mathcal{C}_0^\infty(K)$. Lemma 2.1 implies

$$\frac{1}{|K|}\|\hat{f}\|_2^2 = \lim_{T\to\infty}\frac{1}{2T}\sum_{k=-T}^{T}\bigl|\hat{f}(\tilde{m}_k)\bigr|^2. \tag{6}$$

The right-hand side in (6) is given by

$$c_K \lim_{\epsilon\downarrow 0}\epsilon\sum_{k\in\mathbb{Z}}\bigl|\varphi(\epsilon m_k)\bigr|^2\bigl|\hat{f}(\tilde{m}_k)\bigr|^2 \tag{7}$$

where φ is any function in the Schwartz class $\mathcal{S}(\mathbb{R})$ normalized by $\|\varphi\|_2 = 1$. The constant $c_K = \frac{C}{|K|}$ is taking care of the density of the sequence $m_k, k \in \mathbb{Z}$ and C only depends on the lattice Γ. At this stage we use the auxiliary function of the real variable t defined as

$$F_\epsilon(t) = \sqrt{\epsilon}\sum_{k\in\mathbb{Z}}\varphi(\epsilon m_k)\hat{f}(\tilde{m}_k)\exp(2\pi \mathrm{i} m_k t). \tag{8}$$

We denote by ϕ the Fourier transform of φ. We will suppose that $\phi \in \mathcal{C}_0^\infty([-1,1])$ is a positive and even function. Since $|I| > \operatorname{dens} M_K$, Beurling's theorem applies to the interval I, to the set of frequencies M_K and to the trigonometric sum defined in (8). Then one has

$$\epsilon\sum_{k\in\mathbb{Z}}\bigl|\varphi(\epsilon m_k)\bigr|^2\bigl|\hat{f}(\tilde{m}_k)\bigr|^2 \leqslant C\int_I \bigl|F_\epsilon(t)\bigr|^2\,\mathrm{d}t. \tag{9}$$

Let us compute the lim sup as $\epsilon \to 0$ of the right-hand side of (9). To this aim, we use the definition of M_K and write

$$F_\epsilon(t) = \sqrt{\epsilon}\sum_{\gamma^*\in\Gamma^*}\varphi\bigl(\epsilon p_2(\gamma^*)\bigr)\hat{f}\bigl(p_1(\gamma^*)\bigr)\exp\bigl(2\pi \mathrm{i} p_2(\gamma^*)t\bigr). \tag{10}$$

Poisson identity says that this sum can be computed on the dual lattice. We then have

$$F_\epsilon(t) = c(\Gamma)\frac{1}{\sqrt{\epsilon}}\sum_{\gamma\in\Gamma}\phi\left(\frac{t-p_2(\gamma)}{\epsilon}\right)f\bigl(p_1(\gamma)\bigr). \tag{11}$$

We then return to the estimation of

$$\limsup_{\epsilon\downarrow 0}\int_I \bigl|F_\epsilon(t)\bigr|^2\,\mathrm{d}t, \tag{12}$$

where F_ϵ is given by (11). To this end, we notice that all terms in the right-hand side of (11) for which $|p_1(\gamma)| \geqslant \alpha + \epsilon$ vanish on $I = [-\alpha, \alpha]$. Indeed the support of ϕ is contained in $[-1,1]$. We can restrict the summation to the set $\Lambda_{I,\epsilon} = \{p_1(\gamma);\ \gamma \in \Gamma,\ |p_2(\gamma)| \leqslant \alpha + \epsilon\}$. For $0 \leqslant \epsilon \leqslant 1$ we have

$$\lim_{\epsilon\to 0}\Lambda_{I,\epsilon} = \Lambda_I \quad \text{and} \quad \Lambda_{I,\epsilon} \subset \Lambda_{I,1}. \tag{13}$$

We split F_ϵ into a sum $F_\epsilon = F_\epsilon^N + R_N$ where

$$F_\epsilon^N(t) = \frac{1}{\sqrt{\epsilon}}\sum_{\gamma\in\Gamma,\,|p_1(\gamma)|\leqslant N,\,|p_2(\gamma)|\leqslant\alpha+\epsilon}\phi\left(\frac{t-p_2(\gamma)}{\epsilon}\right)f\bigl(p_1(\gamma)\bigr), \tag{14}$$

and

$$R_N(t) = \frac{1}{\sqrt{\epsilon}}\sum_{\gamma\in\Gamma,\,|p_1(\gamma)|>N,\,|p_2(\gamma)|\leqslant\alpha+\epsilon}\phi\left(\frac{t-p_2(\gamma)}{\epsilon}\right)f\bigl(p_1(\gamma)\bigr). \tag{15}$$

The triangle inequality yields $\|R_N\|_2 \leqslant \epsilon_N \|\phi\|_2$ with

$$\epsilon_N = \sum_{\gamma\in\Gamma,\,|p_1(\gamma)|>N,\,|p_2(\gamma)|\leqslant\alpha+1} \left|f\big(p_1(\gamma)\big)\right|. \tag{16}$$

Let us observe that this series converges. Therefore ϵ_N tends to 0. Indeed f belongs to the Schwartz class and the set $Y = \{p_1(\gamma);\ |p_2(\gamma)| \leqslant \alpha + 1\}$ is uniformly sparse in $\mathbb{R}^n$. Using the terminology of [3], Y is a "model set". For the term (14) the estimations are more involved. Since $|p_1(\gamma)| \leqslant N$, the points $p_2(\gamma)$ appearing in (14) are separated by a distance $\geqslant \beta_N > 0$. If $0 < \epsilon < \beta_N$ the different terms in (14) have disjoint supports which implies

$$\left\|F_\epsilon^N\right\|_{L^2(I)} \leqslant \sigma(N,\epsilon)\|\phi\|_2 \tag{17}$$

where

$$\sigma(N,\epsilon)^2 = \sum_{\gamma\in\Gamma,\,|p_1(\gamma)|\leqslant N,\,|p_2(\gamma)|\leqslant\alpha+\epsilon} \left|f\big(p_1(\gamma)\big)\right|^2.$$

If ϵ is small enough we have

$$\left\{\gamma\in\Gamma,\ \left|p_1(\gamma)\right|\leqslant N,\ \left|p_2(\gamma)\right|\leqslant\alpha+\epsilon\right\} = \left\{\gamma\in\Gamma,\ \left|p_1(\gamma)\right|\leqslant N,\ \left|p_2(\gamma)\right|\leqslant\alpha\right\}$$

and $\sigma(N,\epsilon) = \sigma(N,0)$. Therefore

$$\limsup_{\epsilon\to 0} \int_I \left|F_\epsilon(t)\right|^2 \mathrm{d}t \leqslant \sum_{\lambda\in\Lambda_I} \left|f(\lambda)\right|^2 + \eta_N \tag{18}$$

and letting $N \to \infty$ we obtain the first claim.

The proof of the second claim uses the same strategy and notations. The first assertion of Beurling's theorem is used and the details can be found in a forthcoming paper.

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