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Dynamical bounds for Sturmian Schrödinger operators

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Abstract

The Fibonacci Hamiltonian, that is a Schrödinger operator associated to a Sturmian potential with respect to the golden number has been investigated intensively in recent years. Damanik and Tcheremchantsev developed a method and found a non-trivial dynamical upper bound for transport exponents for this model. This method can be generalized to obtain results for almost all irrational numbers. As a counter example, we exhibit a pathological irrational number with no possible better bound. Moreover, we establish a global lower bound for the lower box dimension of the spectrum that could be used to obtain a dynamical lower bound for irrational numbers with bounded density. ***To cite this article: L. Marin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Bornes dynamiques pour un opérateur Sturmien. Le modèle de Fibonacci, c'est-à-dire un opérateur de Schrödinger associé à un potentiel Sturmien dépendant du nombre d'Or, a fait l'objet de nombreuses études ces dernières années. Cette note a pour objet de généraliser les résultats obtenus par Damanik et Tcheremchantsev sur les exposants de transport du modèle de Fibonacci au même opérateur associé à d'autres nombres irrationnels. Avec leur méthode, nous donnons une borne dynamique supérieure pour presque tout nombre irrationnel. Nous donnons un contre exemple pour lequel aucune nouvelle borne n'est possible. Enfin nous donnons une borne inférieure pour la dimension de boîte du spectre de l'opérateur. Cela nous donnera pour les nombres irrationnels à densité bornée une borne dynamique inférieure pour les exposants de transports. ***Pour citer cet article : L. Marin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

On considère H_β un opérateur de Schrödinger discret et unidimensionnel défini par (1) et (2). Ce modèle a été étudié grandement au cours des dernières années, en particulier le cas Fibonacci, où β est le nombre d'or. On recherche des propriétés spectrales et dynamiques. Les deux sont liées, ainsi le degré de continuité de la mesure spectrale minore les exposants de croissance des probabilités extérieures (voir [6,3]). Nous nous intéressons ici essentiellement à la dynamique.

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On note les probabilités extérieures en moyenne dans le temps

$$P(N,T)=\sum_{|n|>N} a(n,T)$$

avec

$$a(n,T)=\frac{2}{T}\int_0^\infty \mathrm{e}^{-2t/T}\left|\left\langle \mathrm{e}^{-\mathrm{i}tH}\delta_1,\delta_n\right\rangle\right|^2\,\mathrm{d}t.$$

La quantité $a(n,T)$ correspond à la probabilité que le système se trouve à la position n avant le temps T. Les probabilités extérieures sont la somme de ces probabilités. Pour tout $\alpha\in[0,+\infty]$, voir [4]

$$S^-(\alpha)=-\liminf_{T\to\infty}\frac{\log P(T^\alpha-2,T)}{\log T},\qquad S^+(\alpha)=-\limsup_{T\to\infty}\frac{\log P(T^\alpha-2,T)}{\log T}.$$

On est particulièrement interessé par les exposants critiques suivant, appelés exposants de transport :

$$\alpha_l^\pm=\sup\{\alpha\geqslant 0\colon\ S^\pm(\alpha)=0\},\qquad \alpha_u^\pm=\sup\{\alpha\geqslant 0\colon\ S^\pm(\alpha)<\infty\}.$$

On peut interpréter $\alpha_l^\pm$ comme le taux (inférieur et supérieur) de propagation de la partie essentielle du paquet d'ondes et $\alpha_u^\pm$ comme le taux de propagation de la partie la plus rapide (i.e. polynomialement petite) du paquet d'ondes. En particulier, pour $\alpha>\alpha_u^+$, $P(T^\alpha,T)$ décroit vers 0 plus vite que n'importe quelle puissance inverse de T. On a toujours $0\leqslant\alpha_l^\pm\leqslant\alpha_u^\pm$. On a en plus pour ce type de modèle $\alpha_u^\pm\leqslant 1$. Cette borne est appelée borne balistique et est la propagation la plus rapide possible du paquet d'ondes. Pour certains cas particuliers, on peut obtenir une meilleure borne, i.e. plus petite que 1, notamment pour le cas Fibonacci dans [2]. Cette note étend ce résultat à presque tous les nombres irrationnels pour le modèle Sturmien.

On rappelle la suite des réduites associée à β :

$$q_{k+1}=a_{k+1}q_k+q_{k-1},\quad q_0=1,\quad q_{-1}=0.$$

Théorème 0.1. *Soit β un nombre irrationnel et H_β défini avec* (1), (2) *et $V>20$. Si $D=\limsup_k\frac{\log q_k}{k}$ est fini alors*

$$\alpha_u^+\leqslant\frac{2D}{\log\frac{V-8}{3}}.$$

De plus, pour un nombre dont le développement en fraction continue ne contient pas de 1*, la borne dynamique devient*

$$\alpha_u^+\leqslant\frac{D}{\log\frac{V-8}{3}}.$$

Lemme 0.2. *(Khintchin.) Pour Lebesgue presque tout β,*

$$D=\limsup_k\frac{\log q_k}{k}=D_K=\frac{\pi^2}{12\log 2},\qquad M=\liminf_k(a_1\cdots a_k)^{1/k}=M_K=2.685\ldots,$$

$$C=\limsup_k\frac{3}{k}\sum_{j=1}^k\log(a_j+2)\leqslant 3(\log M_K+\log 2)=5.04\ldots.$$

Corollaire 0.3. *Pour Lebesgue presque tout β, on a*

$$\alpha_u^+\leqslant\frac{2D_K}{\log\frac{V-8}{3}}.$$

Corollaire 0.4. *Pour un nombre précieux $\beta=[a,a,a,a,\ldots]$, $a\neq 1$, la borne devient*

$$\alpha_u^+\leqslant\frac{\log(a+\beta)}{\log\frac{V-8}{3}}.$$

Remarque 1. On remarque que pour V assez grand, les théorèmes donnent une borne plus petite que 1. La preuve qui suit celle de [7] utilise le résultat de [8] qui nécessite l'hypothèse $V > 20$. Le théorème suivant nous dit que cette situation ne peut s'étendre au cas $D = +\infty$.

Théorème 0.5. *Il existe un nombre irrationnel ω avec $D = +\infty$ tel que pour tout $V > 20$.*

$$\alpha_u^+ = 1.$$

Nous donnons ici une borne inférieure pour la dimension de boîte du spectre de l'opérateur et la borne dynamique inférieure qui en découle (voir [5,11]).

Théorème 0.6. *On note $C_k = \frac{3}{k}\sum_{j=1}^k \log(a_j + 2)$. Pour un nombre irrationnel β, avec $C = \liminf_k C_k < +\infty$, on a*

$$\dim_B^{\pm}(\sigma) \geqslant \frac{1}{2}\frac{\log 2}{C + \log(V+5)}.$$

On peut montrer avec β un nombre irrationnel à densité bornée i.e. vérifiant $\limsup \frac{1}{n}\sum_{k=1}^n a_k < +\infty$ que la dimension de boîte du spectre minore α_l^-.

Théorème 0.7. *Pour un nombre irrationnel β à densité bornée, on a*

$$\alpha_u^- \geqslant \frac{1}{2}\frac{\log 2}{C + \log(V+5)}$$

avec $C = \limsup \frac{3}{k}\sum_{j=1}^k \log(a_j + 2)$.

1. Introduction

We consider the one dimensional discrete Schrödinger operator H_β,

$$[H_\beta \psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n) \tag{1}$$

acting on $l^2(\mathbb{Z})$, associated to a Sturmian potential $V(n)$ given by

$$V(n) = \left(\lfloor (n+1)\beta \rfloor - \lfloor n\beta \rfloor\right)V \tag{2}$$

with β an irrational number in $[0,1]$ and V a positive constant. We assume that $V > 20$. Proofs require a previous result in [8] which needs that assumption. We denote the continued fraction expansion of β by

$$\beta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_1, a_2, \ldots].$$

The Fibonacci model, H_β with $\beta = \frac{\sqrt{5}-1}{2}$ is the most studied example in Sturmian model. For this kind of operators, $\mathrm{e}^{-\mathrm{itH}}\delta_1$ is known to spread out (over the canonical basis $\{\delta_n\}_{n\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$). The spectral properties and dynamics of this class of operators are of interest. Dynamical field consists in find out bounds controlling the spreading. These two kinds of properties are connected: the continuity degree of the spectral measure bounds from below the spreading. See [11] for last results. We are interested in dynamical bounds. We denote the time average outside probabilities

$$P(N,T) = \sum_{|n|>N} a(n,T),$$

with

$$a(n,T) = \frac{2}{T}\int_0^\infty \mathrm{e}^{-2t/T}\left|\left\langle \mathrm{e}^{-\mathrm{i}tH}\delta_1, \delta_n\right\rangle\right|^2 \mathrm{d}t.$$

The time average probabilities $a(n,T)$ measure the probability for the system to be in position n before time T, $P(N,T)$, as the sum of them, sizes the probability to be outside the ball of radius N at time T.

For all $\alpha \in [0,+\infty]$, see [4]

$$S^-(\alpha) = -\liminf_{T\to\infty} \frac{\log P(T^\alpha - 2, T)}{\log T}, \qquad S^+(\alpha) = -\limsup_{T\to\infty} \frac{\log P(T^\alpha - 2, T)}{\log T}.$$

We have $0 \leqslant S^\pm(\alpha) \leqslant +\infty$. The following critical exponents particularly interesting, are called transport exponents:

$$\alpha_l^\pm = \sup\{\alpha \geqslant 0\colon\ S^\pm(\alpha) = 0\}, \qquad \alpha_u^\pm = \sup\{\alpha \geqslant 0\colon\ S^\pm(\alpha) < \infty\}.$$

They verify $0 \leqslant \alpha_l^\pm \leqslant \alpha_u^\pm$. In particular, if $\gamma > \alpha_u^+$ then $P(T^\gamma, T)$ goes to 0 faster than any inverse power of T. One can interpret $\alpha_l^\pm$ as the (lower and upper) rates of propagation of the essential part of the wavepacket and $\alpha_u^\pm$ as the rates of propagation of the fastest polynomially small part of the wavepacket.

Moreover, we always have for this kind of models $\alpha_u^+ \leqslant 1$. This upper bound, called ballistic, is the fastest rate of spreading of the wavepacket.

In particular cases, one can find $\alpha_u^+ < 1$, that is a non trivial dynamical upper bound. Results are recent for Fibonacci case (see [2]). Our main statement is to generalize this approach to almost every irrational numbers. We give also a new lower bound for the box dimension of the spectrum which implies a dynamical lower bound.

2. Statements of the results

For any $\beta \in [0,1]$, define the sequence

$$q_{k+1} = a_{k+1}q_k + q_{k-1}, \quad q_0 = 1, \quad q_{-1} = 0.$$

Theorem 2.1. *Let β be an irrational number. Let H_β be defined as in* (1) *with the Sturmian potential associated to β. Suppose $V > 20$. If $D = \limsup_k \frac{\log q_k}{k}$ is finite then*

$$\alpha_u^+ \leqslant \frac{2D}{\log(\frac{V-8}{3})}.$$

Moreover, for an irrational number whose continued fraction expansion contains no 1*, the dynamical upper bound becomes*

$$\alpha_u^+ \leqslant \frac{D}{\log(\frac{V-8}{3})}.$$

We recall now an interesting probability result:

Lemma 2.2. *(Khinchin.) For almost all β with respect to Lebesgue measure,*

$$D = \limsup_k \frac{\log q_k}{k} = D_K = \frac{\pi^2}{12\log 2}, \qquad M = \liminf_k (a_1 \ldots a_k)^{1/k} = M_K = 2.685\ldots,$$

$$C = \limsup_k \frac{3}{k}\sum_{j=1}^k \log(a_j + 2) \leqslant 3(\log M_K + \log 2) = 5.04\ldots.$$

Corollary 2.3. *For almost every irrational number β with respect to Lebesgue measure, we have*

$$\alpha_u^+ \leqslant \frac{2D_K}{\log(\frac{V-8}{3})}.$$

Corollary 2.4. *For a precious number, that is $\beta = [a,a,a,a,\ldots]$, $a \neq 1$ the bound becomes*

$$\alpha_u^+ \leqslant \frac{\log(a+\beta)}{\log(\frac{V-8}{3})}.$$

Remark 1. For V large enough, the theorem gives a non-trivial upper bound that is smaller than 1.

The statements above hold if $D < +\infty$. In the case $D = +\infty$, we exhibit in the next statement a counter example as it is done in [7].

Theorem 2.5. *There exists an irrational number with $D = +\infty$ such that for any $V > 20$, $\alpha_u^+ = 1$.*

Theorem 2.6. *We have for any irrational number β verifying $C < +\infty$:*

$$\dim_B^{\pm}(\sigma) \geqslant \frac{1}{2}\frac{\log 2}{C + \log(V+5)} \tag{3}$$

where σ is the spectrum of H_β and $\dim_B$ denote the box dimension.

Remark 2. The former bound for box dimension provided in [8] was

$$\dim_B^{\pm}(\sigma) \geqslant \dim_H(\sigma) \geqslant \max\left\{\frac{\log 2}{10\log 2 + 3\log(4(V+8))}, \frac{\log M - \log 3}{\log M + \log(12(V+8))}\right\}.$$

For almost all irrational numbers, that is with M equal to the Khinchin constant (2.685...), our bound is better than above and for every $V > 20$. On the other hand, for any fixed V, one has no improvement with some specific numbers. Taking $\beta = [c, c, c, \ldots]$ implies $M = c$. The bound above goes to 1 and (3) to 0 when c tends to infinity.

A lower bound for box dimension of the spectrum can be relevant to obtain a lower bound for dynamical exponent α_u^- (see [5,11]).

Theorem 2.7. *For irrational numbers with bounded density i.e. verifying $\limsup \frac{1}{n}\sum_{k=1}^n a_k < +\infty$, we have*

$$\alpha_u^- \geqslant \frac{1}{2}\frac{\log 2}{C + \log(V+5)}$$

with $C = \limsup \frac{3}{k}\sum_{j=1}^k \log(a_j + 2)$.

3. Proof of Theorem 2.1 (sketch)

To study spectrum, one considers the free equation $H\psi = z\psi$. As ψ is one dimensional, one can rewrite in a matricial way.

$$\begin{pmatrix} \psi(q_n+1) \\ \psi(q_n) \end{pmatrix} = M(n,z)\begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}.$$

The so-called transfer matrix $M(n,z)$ transfers site 1 to site q_n. Denote $x_n(z)$ the trace of $M(n,z)$. The spectrum of H_β is the limit of the sequence of spectra σ_n of H_{β_n} where H_{β_n} is the periodic operator defined by substituting β by $\beta_n = \frac{q_n}{q_{n+1}}$. It is shown in [10,1] that $\sigma_n = \{E \in \mathbb{R}, |x_n(E)| \leqslant 2\}$. If the sequence $\{x_n(E)\}_n$ is bounded, then E lies in the spectrum of H_β and if E is not in the spectrum, then $\{x_n(E)\}_n$ is not bounded and grows super exponentially (see [10] and [9] for details). The main idea is to consider $\tilde{\sigma}_n$ for $E \in \mathbb{C}$ and not only in $\mathbb{R}$. The set σ_n is made of q_n segments of $\mathbb{R}$. Each band length is bounded from above and from below by decreasing to 0 (as n grows) constants depending of V, n and β [8]. By the Koebe distortion theorem, we can show that the height of $\tilde{\sigma}_n$ is up to a constant the same as length. The proof uses Theorems 1 and 7 in [2] which connect transfer matrix traces with the outside probabilities:

$$P(N,T) \lesssim \exp(-cN) + T^3 \int_{-V-2}^{V+2} \left(\max_{1\leqslant n\leqslant N}\left\| x_n\left(E + \frac{i}{T}\right)\right\|^2\right)^{-1} \mathrm{d}E.$$

Notice the argument of transfer matrix traces is now in the complex plane. For T fixed, we choose explicitly $N(T)$ such that $(E + \frac{i}{T}) \in \tilde{\sigma}_{N(T)}^c$. It is possible since the imaginary part $\frac{1}{T}$ is fixed and as N grows, the height of $\tilde{\sigma}_{N(T)}$ decays to 0. It implies that $x_{N(T)}(E + \frac{i}{T})$ grows super exponentially and that $P(N(T),T)$ goes to 0 faster than any inverse power of T.

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