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Mathematical Problems in Mechanics

Explicit reconstruction of a displacement field on a surface by means of its linearized change of metric and change of curvature tensors

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Abstract

Let ω be a simply-connected open subset in $\mathbb{R}^2$ and let $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ be a smooth immersion. If two symmetric matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ of order two satisfy appropriate compatibility relations in ω, then $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are the linearized change of metric and change of curvature tensor fields corresponding to a displacement vector field $\boldsymbol{\eta}$ of the surface $\boldsymbol{\theta}(\omega)$.

We show here that, when the fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are smooth, the displacement vector $\boldsymbol{\eta}(y)$ at any point $\boldsymbol{\theta}(y)$, $y \in \omega$, of the surface $\boldsymbol{\theta}(\omega)$ can be explicitly computed by means of a "Cesàro–Volterra path integral formula on a surface", i.e., a path integral inside ω with endpoint y, and whose integrand is an explicit function of the functions $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ and their covariant derivatives. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Reconstruction explicite d'un champ de déplacements le long d'une surface au moyen de ses tenseurs linéarisés de changement de métrique et de courbure. Soit ω un ouvert simplement connexe de $\mathbb{R}^2$ et soit $\boldsymbol{\theta} : \omega \to \mathbb{R}^2$ une immersion régulière. Si deux champs $(\gamma_{\alpha\beta})$ et $(\rho_{\alpha\beta})$ de matrices symétriques d'ordre deux satisfont des conditions de compatibilité appropriées dans ω, alors $(\gamma_{\alpha\beta})$ et $(\rho_{\alpha\beta})$ sont les champs de tenseurs linéarisés de changement de métrique et de courbure associés à un champ $\boldsymbol{\eta}$ de déplacements de la surface $\boldsymbol{\theta}(\omega)$.

On montre ici que, si les champs $(\gamma_{\alpha\beta})$ et $(\rho_{\alpha\beta})$ sont réguliers, le vecteur déplacement $\boldsymbol{\eta}(y)$ en tout point $\boldsymbol{\theta}(y)$, $y \in \omega$, de la surface $\boldsymbol{\theta}(\omega)$ peut être calculé explicitement au moyen d'une "intégrale de Cesàro–Volterra" le long d'un chemin dans ω d'extrémité y, et dont l'intégrande est une fonction explicite des fonctions $\gamma_{\alpha\beta}$ et $\rho_{\alpha\beta}$ et de leurs dérivées covariantes. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. The Cesàro–Volterra path integral formula in three-dimensional Cartesian coordinates

Latin indices and exponents range in the set $\{1, 2, 3\}$ and the summation convention with respect to repeated Latin indices and exponents is used in conjunction with this rule. The sets of all real matrices of order three and of all real

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symmetric matrices of order three are denoted $\mathbb{M}^3$ and $\mathbb{S}^3$. The *Cartesian coordinates* of a point $x \in \mathbb{R}^3$ are denoted x^i and the partial derivatives $\partial/\partial x^i$ and $\partial^2/\partial x^i \partial x^j$ are denoted ∂_i and ∂_{ij}.

Let Ω be a connected open subset of $\mathbb{R}^3$. Given any vector field $\mathbf{v} = (v_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$, let the *linearized strains* $e_{ij} = e_{ji} \in \mathcal{C}^2(\Omega)$ be defined by

$$e_{ij} := \frac{1}{2}(\partial_j v_i + \partial_i v_j) \quad \text{in } \Omega.$$

Then it is immediately verified that these functions satisfy the *Saint Venant compatibility conditions*:

$$R_{ijk\ell}(\mathbf{e}) := \partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \text{in } \Omega. \tag{1}$$

It is well known that, if in addition the open set Ω is *simply-connected*, and a symmetric matrix field $\mathbf{e} = (e_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3)$ is given, the Saint Venant compatibility conditions $R_{ijk\ell}(\mathbf{e}) = 0$ become also *sufficient* for the *existence* of a vector field $\mathbf{v} = (v_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ that satisfies

$$\frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} \quad \text{in } \Omega. \tag{2}$$

It seems to be much less known (Ref. [7] constitutes an exception) that an explicit solution to this equation is given by the *Cesàro–Volterra path integral formula in Cartesian coordinates* (3) below, so named after Cesàro [1] and Volterra [8].

Theorem 1. *If Ω is connected, a particular solution $\mathbf{v} = (v_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ to Eqs.* (2) *is given for all $x \in \Omega$ by*

$$v_i(x) = \int\limits_{\boldsymbol{\gamma}(x)} \big\{ e_{ij}(y) + \big(\partial_k e_{ij}(y) - \partial_i e_{kj}(y)\big)\big(x^k - y^k\big)\big\}\, \mathrm{d}y^j, \tag{3}$$

where $\boldsymbol{\gamma}(x)$ is any curve of class $\mathcal{C}^1$ joining a given point $x_0 \in \Omega$ to x, or, in vector form,

$$\mathbf{v}(x) = \int\limits_{\boldsymbol{\gamma}(x)} \mathbf{e}(y)\, \mathbf{dy} + \int\limits_{\boldsymbol{\gamma}(x)} (x - y) \wedge \big(\big[\mathbf{CURL}\, \mathbf{e}(y)\big]\, \mathbf{dy}\big), \tag{4}$$

where the matrix curl operator $\mathbf{CURL} : \mathcal{D}'(\Omega; \mathbb{M}^3) \to \mathcal{D}'(\Omega; \mathbb{M}^3)$ *is defined by* (($\varepsilon^{i\ell k}$) denotes here the Cartesian orientation tensor)

$$(\mathbf{CURL}\, \mathbf{e})^i_j := \varepsilon^{i\ell k} \partial_\ell e_{jk} \quad \textit{for any matrix field } \mathbf{e} = (e_{ij}) \in \mathcal{D}'\big(\Omega; \mathbb{M}^3\big).$$

Proof. That a particular solution of (2) is given by (3) follows from direct differentiation. □

2. The Cesàro–Volterra path integral formula in three-dimensional curvilinear coordinates

We now recast in terms of *curvilinear coordinates* the Saint Venant compatibility conditions and the Cesàro–Volterra path integral formula derived in Section 1 in *Cartesian coordinates*. Proofs will be found in [4].

Let Ω be an open subset of $\mathbb{R}^3$ and let there be given an injective immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$.

The three coordinates x^i of a point $x \in \Omega$ are now the *curvilinear coordinates* of the point $\boldsymbol{\Theta}(x)$. Corresponding partial derivatives will be again denoted ∂_i and ∂_{ij}. The Euclidean norm of $\mathbf{a} \in \mathbb{R}^3$, and the Euclidean inner product and vector product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, are respectively denoted $|\mathbf{a}|$, $\mathbf{a} \cdot \mathbf{b}$, and $\mathbf{a} \wedge \mathbf{b}$. For each $x \in \Omega$, the three vectors $\mathbf{g}_i(x) := \partial_i \boldsymbol{\Theta}(x)$ form the *covariant basis* at the point $\boldsymbol{\Theta}(x)$, and the three vectors $\mathbf{g}^j(x)$ defined by $\mathbf{g}_i(x) \cdot \mathbf{g}^j(x) = \delta_i^j$ for all $x \in \Omega$, form the *contravariant basis* at the same point.

For details about the notions of tensors and their covariant derivatives introduced below, see, e.g., [2]. The *covariant derivatives* $v_{j\|i} \in \mathcal{C}^0(\Omega)$ of a vector field $v_i \mathbf{g}^i$ with covariant components $v_i \in \mathcal{C}^1(\Omega)$ are defined by

$$v_{j\|i} := \partial_i v_j - \Gamma_{ij}^k v_k,$$

where $\Gamma_{ij}^k := \mathbf{g}^k \cdot \partial_i \mathbf{g}_j$, denote the *Christoffel symbols*. The *second-order covariant derivatives* $e_{ij\|k\ell} \in \mathcal{C}^0(\Omega)$ of a second-order tensor field with covariant components $e_{ij} \in \mathcal{C}^2(\Omega)$ are defined by

$$e_{ij\|k\ell} := \partial_\ell e_{ij\|k} - \Gamma_{\ell i}^p e_{pj\|k} - \Gamma_{\ell j}^p e_{ip\|k} - \Gamma_{\ell k}^p e_{ij\|p}.$$

Finally, the contravariant components $\varepsilon^{ijk}(x)$ of the *third-order orientation tensor* are defined for all $x \in \Omega$ by $1/\sqrt{g(x)}$ if $\{i,j,k\}$ is an even permutation of $\{1,2,3\}$; by $-1/\sqrt{g(x)}$ if $\{i,j,k\}$ is an odd permutation of $\{1,2,3\}$; and by 0 if at least two indices are equal, where $g(x) := \det(g_{ij}(x))$.

Relations (5) below constitute the *Saint-Venant compatibility conditions in curvilinear coordinates* and the functions $e_{ij} = \frac{1}{2}(\partial_j \mathbf{v} \cdot \mathbf{g}_i + \partial_i \mathbf{v} \cdot \mathbf{g}_j)$ are the *linearized strains in curvilinear coordinates* associated with the displacement field $v_i \mathbf{g}^i$. The proof of Theorem 2 is found in [6].

Theorem 2. *Given any vector field $\mathbf{v} = v_i \mathbf{g}^i$ with covariant components $v_i \in \mathcal{C}^3(\Omega)$, let the tensor field (e_{ij}) be defined by*

$$e_{ij} = e_{ji} := \frac{1}{2}(\partial_j \mathbf{v} \cdot \mathbf{g}_i + \partial_i \mathbf{v} \cdot \mathbf{g}_j) = \frac{1}{2}(v_{i\|j} + v_{j\|i}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma_{ij}^k v_k.$$

Then the covariant components $e_{ij} \in \mathcal{C}^2(\Omega)$ of this tensor satisfy

$$e_{ki\|j\ell} + e_{j\ell\|ki} - e_{jk\|\ell i} - e_{i\ell\|kj} = 0 \quad \text{in } \Omega. \tag{5}$$

Assume that the open set Ω is simply-connected, and let there be given a symmetric tensor field (e_{ij}) whose covariant components $e_{ij} \in \mathcal{C}^2(\Omega)$ satisfy relations (5). *Then conversely, there exists a vector field $\mathbf{v} = v_i \mathbf{g}^i \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ that satisfies*

$$\frac{1}{2}(\partial_j \mathbf{v} \cdot \mathbf{g}_i + \partial_i \mathbf{v} \cdot \mathbf{g}_j) = e_{ij} \quad \text{in } \Omega. \tag{6}$$

Formula (7) below constitutes the *Cesàro–Volterra path integral formula in curvilinear coordinates.* For a proof, see [4].

Theorem 3. *Assume that the open set Ω is connected and simply-connected and let x_0 be a given point in Ω. Then a particular solution to Eqs.* (6) *is given for all $x \in \Omega$ by*

$$v_i(x)\mathbf{g}^i(x) = \int_{\boldsymbol{\gamma}(x)} e_{ij}(y)\mathbf{g}^i(y)\,\mathrm{d}y^j + \int_{\boldsymbol{\gamma}(x)} \big(\boldsymbol{\Theta}(x) - \boldsymbol{\Theta}(y)\big) \wedge \big(\varepsilon^{i\ell k}(y) e_{jk\|\ell}(y)\mathbf{g}_i(y)\,\mathrm{d}y^j\big), \tag{7}$$

where $\boldsymbol{\gamma}(x)$ is any curve of class $\mathcal{C}^1$ joining x_0 to x in Ω.

3. The Cesàro–Volterra path integral formula on a surface

We first recall some classical definitions and notions regarding surfaces defined in terms of *curvilinear coordinates*; for details, see, e.g., [2]. In what follows, Greek indices range in the set $\{1,2\}$, Latin indices range as before in the set $\{1,2,3\}$, and the summation convention with respect to repeated indices also applies to Greek indices.

Let ω be an open subset of $\mathbb{R}^2$. The coordinates of a point $\tilde{x} \in \omega$ are denoted x^α and the partial derivatives $\partial/\partial x^\alpha$ and $\partial^2/\partial x^\alpha \partial x^\beta$ are denoted ∂_α and $\partial_{\alpha\beta}$.

Let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ be an injective immersion. Then the image $S := \boldsymbol{\theta}(\omega)$ is a *surface* immersed in $\mathbb{R}^3$, equipped with x^1 and x^2 as its *curvilinear coordinates*. For each $\tilde{x} \in \omega$, the vectors $\mathbf{a}_\alpha(\tilde{x}) := \partial_\alpha \boldsymbol{\theta}(\tilde{x})$ form the *covariant basis* at the point $\boldsymbol{\theta}(\tilde{x})$, and the tangent vectors $\mathbf{a}^\beta(\tilde{x})$ defined by $\mathbf{a}_\alpha(\tilde{x}) \cdot \mathbf{a}^\beta(\tilde{x}) = \delta_\alpha^\beta$ form the *contravariant basis* of the tangent plane at the same point. A unit normal vector to S at $\boldsymbol{\theta}(\tilde{x})$ is defined by

$$\mathbf{a}_3(\tilde{x}) = \mathbf{a}^3(\tilde{x}) := \frac{\mathbf{a}_1(\tilde{x}) \wedge \mathbf{a}_2(\tilde{x})}{|\mathbf{a}_1(\tilde{x}) \wedge \mathbf{a}_2(\tilde{x})|}.$$

The covariant components of the *first fundamental form* of S, are defined for all $\tilde{x} \in \omega$ by $a_{\alpha\beta}(\tilde{x}) = \mathbf{a}_\alpha(\tilde{x}) \cdot \mathbf{a}_\beta(\tilde{x})$, and its contravariant components are defined by $a^{\alpha\beta}(\tilde{x}) = \mathbf{a}^\alpha(\tilde{x}) \cdot \mathbf{a}^\beta(\tilde{x})$.

The covariant components of the *second fundamental form* of S are defined for all $\tilde{x} \in \omega$ by $b_{\alpha\beta}(\tilde{x}) = -\partial_\alpha \mathbf{a}_3(\tilde{x}) \cdot \mathbf{a}_\beta(\tilde{x}) = \partial_\alpha \mathbf{a}_\beta(\tilde{x}) \cdot \mathbf{a}_3(\tilde{x})$, and its mixed components are defined by $b_\alpha^\tau(\tilde{x}) = -\partial_\alpha \mathbf{a}_3(\tilde{x}) \cdot \mathbf{a}^\tau(\tilde{x}) = \partial_\alpha \mathbf{a}^\tau(\tilde{x}) \cdot \mathbf{a}_3(\tilde{x})$.

The *Christoffel symbols* are defined by $\Gamma^\tau_{\alpha\beta} = \Gamma^\tau_{\beta\alpha} := \frac{1}{2} a^{\tau\nu}(\partial_\alpha a_{\beta\nu} + \partial_\beta a_{\alpha\nu} - \partial_\nu a_{\alpha\beta}) = \mathbf{a}^\tau \cdot \partial_\alpha \mathbf{a}_\beta$.

The *covariant derivatives* $\eta_{\alpha|\beta} \in \mathcal{C}^0(\omega)$ of a first-order tensor field with covariant components $\eta_\alpha \in \mathcal{C}^1(\omega)$ are defined by

$$\eta_{\alpha|\beta} := \partial_\beta \eta_\alpha - \Gamma^\nu_{\beta\alpha} \eta_\nu.$$

The *covariant derivatives* $T_{\alpha\beta|\sigma} \in \mathcal{C}^0(\omega)$ of a second-order tensor field with covariant components $T_{\alpha\beta} \in \mathcal{C}^1(\omega)$ are defined by

$$T_{\alpha\beta|\sigma} := \partial_\sigma T_{\alpha\beta} - \Gamma^\nu_{\sigma\alpha} T_{\nu\beta} - \Gamma^\nu_{\sigma\beta} T_{\alpha\nu}.$$

The *covariant derivatives* $T^\alpha_\sigma|_\tau \in \mathcal{C}^0(\omega)$ of a symmetric second-order tensor field with mixed components $T^\alpha_\sigma \in \mathcal{C}^1(\omega)$ are defined by

$$T^\alpha_\sigma|_\tau := \partial_\tau T^\alpha_\sigma - \Gamma^\mu_{\tau\sigma} T^\sigma_\mu + \Gamma^\alpha_{\tau\mu} T^\mu_\sigma.$$

The *second-order covariant derivatives* $T_{\alpha\beta|\sigma\tau} \in \mathcal{C}^0(\omega)$ of a second-order tensor field with covariant components $T_{\alpha\beta} \in \mathcal{C}^2(\omega)$ are defined by

$$T_{\alpha\beta|\sigma\tau} := \partial_\tau T_{\alpha\beta|\sigma} - \Gamma^\nu_{\tau\alpha} T_{\nu\beta|\sigma} - \Gamma^\nu_{\tau\beta} T_{\alpha\nu|\sigma} - \Gamma^\nu_{\tau\sigma} T_{\alpha\beta|\nu}.$$

Finally, the contravariant components $\varepsilon^{\alpha\beta}(\tilde{x})$ of the *second-order orientation tensor* are defined for all $\tilde{x} \in \omega$ by $1/\sqrt{a(\tilde{x})}$ if $\alpha = 1$ and $\beta = 2$; by 0 if $\alpha = \beta$; and by $-1/\sqrt{a(\tilde{x})}$ if $\alpha = 2$ and $\beta = 1$, where $a(\tilde{x}) := \det(a_{\alpha\beta}(\tilde{x}))$.

We first justify the *Saint Venant compatibility conditions "on a surface"* (cf. (10) below), thus following the same presentation as in the two previous sections:

Theorem 4. *Let ω be an open subset of $\mathbb{R}^2$. Given any vector field $\boldsymbol{\eta} = \eta_i \mathbf{a}^i$ with covariant components $\eta_i \in \mathcal{C}^3(\omega)$, let the tensor field $(\gamma_{\alpha\beta})$ be defined by*

$$\gamma_{\alpha\beta} = \gamma_{\beta\alpha} := \frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}) = \frac{1}{2}(\eta_{\beta|\alpha} + \eta_{\alpha|\beta}), \tag{8}$$

and let the tensor field $(\rho_{\alpha\beta})$ be defined by

$$\begin{aligned}\rho_{\alpha\beta} = \rho_{\beta\alpha} &:= \left(\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\nu_{\alpha\beta}\partial_\nu \boldsymbol{\eta}\right) \cdot \mathbf{a}_3. \\ &= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha \eta_{\sigma|\beta} + b^\tau_\beta \eta_{\tau|\alpha} + b^\tau_{\beta|\alpha}\eta_\tau.\end{aligned} \tag{9}$$

Then the covariant components $\gamma_{\alpha\beta} \in \mathcal{C}^2(\omega)$ and $\rho_{\alpha\beta} \in \mathcal{C}^1(\omega)$ of these two tensors necessarily satisfy

$$\begin{aligned}&\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^\nu_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^\nu_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu} = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha} \quad \text{in } \omega, \\ &\rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^\nu_\sigma(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^\nu_\tau(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu}) \quad \text{in } \omega,\end{aligned} \tag{10}$$

where

$$R^\nu_{\cdot\alpha\sigma\tau} := \partial_\sigma \Gamma^\nu_{\tau\alpha} - \partial_\tau \Gamma^\nu_{\sigma\alpha} + \Gamma^\mu_{\tau\alpha}\Gamma^\nu_{\sigma\mu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\nu_{\tau\mu}.$$

Assume that the open set ω is simply-connected, and let there be given two symmetric tensor fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ whose covariant components $\gamma_{\alpha\beta} \in \mathcal{C}^2(\omega)$ and $\rho_{\alpha\beta} \in \mathcal{C}^1(\omega)$ together satisfy Eqs. (10). *Then there exists a vector field $\boldsymbol{\eta} = \eta_i \mathbf{a}^i$ with covariant components $\eta_i \in \mathcal{C}^3(\omega)$ such that*

$$\frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}) = \gamma_{\alpha\beta} \quad \text{and} \quad \left(\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\nu_{\alpha\beta}\partial_\nu \boldsymbol{\eta}\right) \cdot \mathbf{a}_3 = \rho_{\alpha\beta} \quad \text{in } \omega. \tag{11}$$

Proof. The necessity and sufficiency of the relations (10) have already been established in [3] in a different functional analytic setting, but the computations involved (which are quite lengthy) are formally identical to those needed in the present situation. $\square$

The functions $\gamma_{\alpha\beta}$ defined in (8) and the functions $\rho_{\alpha\beta}$ defined in (9) are respectively the covariant components of the *linearized change of metric* and *linearized change of curvature tensors*, associated with the vector field $\eta_i \mathbf{a}^i$. These tensors play a fundamental role in the theory of *linearly elastic shells*; for details, see, e.g., [2].

We conclude our analysis by deriving, in (12) below, the announced *Cesàro–Volterra path integral formula on a surface*:

Theorem 5. *Assume that the open set ω is connected and simply-connected and let $\tilde{x}_0$ be a given point in ω. Then a particular solution to Eqs.* (11) *is given for all $\tilde{x} \in \omega$ by*

$$\eta_i(\tilde{x})\mathbf{a}^i(\tilde{x}) = \int\limits_{\boldsymbol{\gamma}(\tilde{x})} \gamma_{\alpha\beta}(\tilde{y})\mathbf{a}^\alpha(\tilde{y})\,\mathrm{d}y^\beta + \int\limits_{\boldsymbol{\gamma}(\tilde{x})} \big(\boldsymbol{\theta}(\tilde{x}) - \boldsymbol{\theta}(\tilde{y})\big) \wedge \big(\varepsilon^{\alpha\beta}(\tilde{y})\gamma_{\alpha\sigma|\beta}(\tilde{y})\mathbf{a}_3(\tilde{y})\,\mathrm{d}y^\sigma\big)$$
$$+ \int\limits_{\boldsymbol{\gamma}(\tilde{x})} \big(\boldsymbol{\theta}(\tilde{x}) - \boldsymbol{\theta}(\tilde{y})\big) \wedge \big(\varepsilon^{\alpha\beta}(\tilde{y})\big(\rho_{\alpha\sigma}(\tilde{y}) - b_\alpha^\tau(\tilde{y})\gamma_{\tau\sigma}(\tilde{y}) - b_\sigma^\tau(\tilde{y})\gamma_{\alpha\tau}(\tilde{y})\big)\mathbf{a}_\beta(\tilde{y})\,\mathrm{d}y^\sigma\big), \tag{12}$$

where $\boldsymbol{\gamma}(\tilde{x})$ is any curve of class $\mathcal{C}^1$ joining $\tilde{x}_0$ to $\tilde{x}$ in ω.

Idea of the proof. The key idea is to use a canonical extension of any smooth enough vector field defined on the surface $\boldsymbol{\theta}(\omega)$ into a vector field defined on an appropriate three-dimensional tubular neighborhood of $\boldsymbol{\theta}(\omega)$. The starting point is then the *Cesàro–Volterra path integral formula in curvilinear coordinates* (Theorem 3), first expressed in this tubular neighborhood, then restricted in an appropriate way to the surface $\boldsymbol{\theta}(\omega)$. The corresponding, at times delicate, computations are found in [4]. $\square$

It is worth noticing that, like in Cartesian coordinates, one can show that *the path-independence of the Cesàro–Volterra path integral formula in curvilinear coordinates* (12) *implies that the tensor fields* $(\gamma_{\alpha\beta})$ *and* $(\rho_{\alpha\beta})$ *necessarily satisfy the Saint Venant compatibility conditions "on a surface"* (10); cf. [5].

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