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C. R. Acad. Sci. Paris, Ser. I 346 (2008) 829–832

COMPTES RENDUS MATHEMATIQUE

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Ordinary Differential Equations

Sectorial normalization of Poisson structures

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Received 29 May 2008; accepted after revision 3 July 2008

Presented by Bernard Malgrange

Abstract

Our study deals with some singular Poisson structures holomorphic near $0 \in \mathbb{C}^n$ admitting a polynomial normal form, i.e. a finite number of formal invariants. Their normalizing series generally diverge. We show the existence of normalizing transformations holomorphic on some sectorial domains $a < \arg x^R < b$, where x^R denotes a monomial associated to the problem. It follows an analytic classification. ***To cite this article: P. Lohrmann, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Normalisation sectorielle de structures de Poisson. Notre étude porte sur une catégorie de structures de Poisson singulières holomorphes au voisinage de $0 \in \mathbb{C}^n$ et admettant une forme normale formelle polynomiale i.e. un nombre fini d'invariants formels. Les séries normalisantes sont divergentes en général. On montre l'existence de transformations normalisantes holomorphes sur des domaines sectoriels de la forme $a < \arg x^R < b$, où x^R est un monôme associé au problème. Il s'ensuit une classification analytique. ***Pour citer cet article : P. Lohrmann, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

In this Note we view a Poisson structure as a bivector field

$$\Pi = \frac{1}{2}\sum_{1 \neq j} \pi_{ij}(x)\partial/\partial x_i \wedge \partial/\partial x_j$$

on $\mathbb{C}^n$, satisfying the relation $[\Pi, \Pi] = 0$, where $[\cdot,\cdot]$ is the Schouten–Nijenhuis bracket (natural extension of the Lie bracket cf. [2]). Denote by $X_\Pi(f) := [x^I, \Pi]$ the Hamiltonian vector field of a function f with respect to the Poisson structure Π. We are interested in holomorphic Poisson structures vanishing at $0 \in \mathbb{C}^n$, with a vanishing linear part and a quadratic part satisfying some generic hypothesis (H) (cf. [1]). These Poisson structures take the form

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doi:10.1016/j.crma.2008.07.007

$\Pi = \Pi_2 + \text{h.o.t.}$, where $\Pi_2 = \frac{1}{2}\sum_{i\neq j} a_{ij}x_ix_j\partial/\partial x_i \wedge \partial/\partial x_j$, $a_{ij} \in \mathbb{C}$, denotes the quadratic part, and h.o.t. denotes higher order terms. In [1] is proved the existence of formal transformations $\hat{\Phi}$ such that

$$\hat{\Phi}^*(\Pi) = \Pi_2 + \sum_{A.I=0} x^I \Pi^I.$$

Here $A.I = 0$ means that the Hamiltonian vector field of x^I with respect to Π_2 vanishes, i.e. the Lie derivatives of x^I along the vector fields $\sum_{j=1}^n a_{ij}x_j\partial/\partial x_j$ vanish. The bivector fields Π^I take the form $\Pi^I = \sum_{i,j} a_{ij}^I x_ix_j\partial/\partial x_i \wedge \partial/\partial x_j$ with $a_{ij}^I \in \mathbb{C}$. Then we say that $\hat{\Phi}^*(\Pi)$ is under *normal form in the sense of Dufour–Wade* and call monomials $x^I\Pi^I$ with $A.I = 0$ *resonant terms*. In [3] the author of this Note shows that the transformations $\hat{\Phi}$ normalizing Π converge under some algebraic condition (C) on the formal normal form and under a diophantine small divisor condition (D) associated to Π_2. In this note we assume that condition (C) is not satisfied. Then the normalizing series generally diverge (cf. [4]).

2. Principal result

Let be $R \in \mathbb{N}^n$, $r, r' > 0$, $\delta \in \mathbb{C}^*$ and

$$DS_s^R(r,r',\delta) = \left\{ y \in \mathbb{C}^n \;\middle|\; \left|\arg y^R - \left(\arg\delta + \pi\left(s + \frac{1}{2}\right)\right)\right| < \pi - \varepsilon,\ 0 < |y^R| < r,\ |y_i| < r' \right\},$$

$s = 0, 1$. Consider a Poisson structure Π having a vanishing 1-jet and a quadratic part Π_2 (cf. introduction) which satisfies the following assumptions:

(P1) Π_2 is 1-*resonant*: any monomial x^I such that $A.I = 0$ is some power of a monomial x^R, called "resonance generator";
(P2) The quadratic part Π_2 is a wedge product of two linear diagonal vector fields S_1 and S_2;
(P3) The quadratic part Π_2 satisfies the diophantine small divisor condition $-\sum_{k>0}\ln(\omega_k)/2^k < \infty$, where $\omega_k := \min\{\beta^I \mid \beta^I \neq 0,\ 1 \leqslant |I| \leqslant 2^k\}$ with $\beta^I = \max_{1\leqslant i\leqslant n}|\beta_i^I|$ and $\beta_i^I := \sum_{j=1}^n a_{ij}I_j$;
(P4) The resonant term of smallest degree takes the form $x^R\Pi^R$, where $\Pi^R = S \wedge C$, with $S \in \mathrm{Vect}(S_1, S_2)$, and C a linear diagonal vector field satisfying $\mathcal{L}_C(x^R) \neq 0$;
(P5) All coordinates of R are strictly positive;
(P6) Write $S_1 = s_1Y_1 + \cdots + s_nY_n$ and $S_2 = s_1'Y_1 + \cdots + s'nY_n$, where $Y_i := x_i\partial/\partial x_i$. Assume that all coefficients s_i, s_i' lie on the same line in the complex plane going through the origin;
(P7) Consider $(\mu_i)_{1\leqslant i\leqslant n}$, $(\alpha_i)_{1\leqslant i\leqslant n}$ defined by $\frac{1}{x^{\Lambda_0}}X_{[\Pi_2 + x^R\Pi^R]}(x^{\Lambda_0}) := \sum_{i=1}^n x_i(\mu_i + \alpha_i x^R)\partial_i$. Assume there exists $\Lambda_0 := (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that there is an index i_0 with $\mu_{i_0} \neq 0$ and $\min_{i\neq i_0}\mathrm{Re}(\frac{\alpha_i}{\beta} - \frac{\mu_i}{\mu_{i_0}}\frac{\alpha_{i_0}}{\beta}) > 0$ (the vector field $\frac{1}{x^\Lambda}X_{[\Pi_2 + x^R\Pi^R]}(x^\Lambda)$ does not depend on the determination of x^Λ).
Let $\delta \in \mathbb{C}$, $|\delta| = 1$ be such that any number $\mu_i(\delta(\sum_{j=1}^n R_j\alpha_j))^{-1}$ is real.

Theorem *(sectorial normalization). Let be Π as in* (P1)–(P7). *Then there exists a set $\mathfrak{A} \subset \mathbb{C}^n$ which is the intersection of an open set and a full-Lebesgue measure set, and depends only on Π_2 and Π^R, such that for any $\Lambda \in \mathfrak{A}$ the following properties hold: there exists an analytic unit v_Λ (unique), $r, r' > 0$, and holomorphic transformations Φ_s^Λ, $s = 0, 1$, (unique) defined on the sectorial domains $DS_s^R(r, r', \delta)$, $s = 0, 1$, such that*

(i) *Φ_s^Λ conjugate $v_\Lambda\Pi$ (which is a Poisson structure) to the polynomial normal form $\Pi^2 + x^R\Pi^R$;*
(ii) *the equalities $x^\Lambda \circ \Phi_s^\Lambda = x^\Lambda$ and $x^R \circ \phi_s^\Lambda = x^R$ hold.*

The transformations Φ_s^Λ, $s = 0, 1$, admit $\hat{\Phi}$ as asymptotic development in the sense of Gérard–Sibuya in the monomial x^R on $DS_s(r, r', \delta)$, where $\hat{\Phi}$ is the unique formal power serie such that $(\hat{\Phi})^*(v_\Lambda\Pi) = \Pi^2 + x^R\Pi^R$ and such that we have $x^\Lambda \circ \hat{\Phi} = x^\Lambda$, $x^R \circ \hat{\phi} = x^R$.

Sketch of the proof. (For a complete proof of the theorem – cf. [4].) Let D be the $\mathbb{C}[x]$-module generated by the set of quadratic diagonal vector fields, i.e. constituted of sums $\sum_{I\in\mathbb{N}^n} x^I \Pi^I$ where I does not have components taking the value -1, and each Π^I denotes a quadratic diagonal bivector field, i.e. a sum of wedge products of linear diagonal vector fields. We show the following result:

Proposition *(FN). Consider $\Pi := \Pi_2 + x^R \Pi^R + \cdots \in D$ satisfying assumptions* (P1) *and* (P4) *of the theorem. Then for any $\Lambda \in (C^*)^n$ which does not belong to some countable union of hyperplanes of $\mathbb{C}^n$ (not precised here), any formal transformation which conjugates the vector field $\frac{1}{x^\Lambda} X_\Pi(x^\Lambda)$ to the polynomial normal form $\frac{1}{x^\Lambda} X_{[\Pi_2+x^R\Pi^R]}(x^\Lambda)$ also conjugates Π to $\Pi_n := \Pi_2 + x^R \Pi^R$.*

Note that the association $\Pi \to \frac{1}{x^\Lambda} X_\Pi(x^\Lambda)$ is not *intrinsic*.

In [7] Stolovitch considers holomorphic vector fields $X = S + x^R V + \text{h.o.t.}$, where S denotes a 1-resonant linear part S with resonance-generator x^R, and V is a linear diagonal vector field such that $\mathcal{L}_V(x^R) \neq 0$. These vector fields are formally conjugated to the polynomial normal form $S + x^R V$. Stolovitch gives sufficient conditions insuring the existence of holomorphic transformations (generally having divergent power series as asymptotic development) defined on the sectorial domains $DS_s^R(r, r', \delta)$, $s = 0, 1$. This result, which we abbreviate here by (SN), deals with vector fields under some *well prepared* form. In order to transform X into well prepared form, one first applies a transformation holomorphic on a neighborhood of $0 \in \mathbb{C}^n$ which is called *prenormalizing transformation*. One then divides by some suitable analytic unit in order to have all terms other that $x^R V$ tangent to the variety $x^R = 0$. The proof of the existence of prenormalizing transformations given in [6] is based on fine estimations of the number of small divisors. By transposing this result to Poisson structures one obtains a transformation Ψ holomorphic in $0 \in \mathbb{C}^n$ such that $\Psi^*(\Pi) \in D$ and such that the associated vector fields $\frac{1}{x^\Lambda} X_\Pi(x^\Lambda)$ are prenormalized.

For the problem of sectorial normalization of Poisson structures as in Proposition (FN) one can thus take as starting point a Poisson structure Π belonging to D such that the vector fields $\frac{1}{x^\Lambda} X_\Pi(x^\Lambda)$ are prenormalized. Division by some appropriate analytic unit υ_Λ makes the associated vector field $\upsilon_\Lambda \frac{1}{x^\Lambda} X_\Pi(x^\Lambda)$ well prepared. By Theorem (SN) there are two holomorphic transformations Φ_s^Λ normalizing the vector field $\upsilon_\Lambda \frac{1}{x^\Lambda} X_\Pi(x^\Lambda)$ on $DS_s^R(r, r', \delta)$, $s = 0, 1$. It then follows from Proposition (FN) that $\Phi_s^*(\upsilon_\Lambda \Pi)$ takes the form $\Pi_2 + x^R \Pi^R + \mathcal{R}_s^\Lambda$, with an infinite flat rest $\mathcal{R}_s^\Lambda$, holomorphic on $DS_s^R(r, r', \delta)$. The main difficulty then is to show that $\mathcal{R}_s^\Lambda = 0$, i.e. that Φ_s^Λ conjugates Π to the normal form $\Pi_n = \Pi_2 + x^R \Pi^R$ on $DS_s^R(r, r', \delta)$, $s = 0, 1$. Theorem (SN) states in addition that Φ_s^Λ admits $\hat{\Phi}^\Lambda$ as asymptotic development in the sense of Gérard–Sibuya. It follows that

$$\mathcal{R}_s^\Lambda = \frac{1}{2} \sum_{\substack{i\neq j\\ I\in\mathbb{D}}} x^I f_{ijs}^{I,\Lambda}(x^R) Y_i \wedge Y_j, \tag{1}$$

where $\mathbb{D} \subset \mathbb{N}^n$ is the set of multi-indexes such that x^R does not divide x^I, and $f_{ijs}^{I,\Lambda}$ are infinitely flat functions defined on $DS_s^R(r, r', \delta)$.

Consider for a moment the formal case. In the proof of Proposition (FN) one assumes $\Pi = \Pi_2 + x^R \Pi^R + x^{I_0} \Pi^{I_0} + \cdots$, where $x^{I_0} \Pi^{I_0}$ is a term of smallest degree which is non-resonant. By a computation one deduces from the compatibility relation $[\Pi_2, x^{I_0} \Pi^{I_0}] = 0$ that the vector field $\frac{1}{x^\Lambda} X_{[x^{I_0}\Pi^{I_0}]}(x^\Lambda)$ does not vanish. The idea for proving that $\mathcal{R}_s^\Lambda$ vanishes is to make the *same* computation in the case $(\Phi_s^\Lambda)^*(\upsilon_\Lambda \Pi) = \Pi_2 + x^R \Pi^R + \mathcal{R}_s^\Lambda$, with $\mathcal{R}_s^\Lambda$ as in (1), i.e.

$$\left(\Phi_s^\Lambda\right)^*(\upsilon_\Lambda \Pi) = \Pi_2 + x^R \Pi^R + x^{I_0} \Pi_s^{I_0,\Lambda} + \cdots,$$

here $I_0 \in \mathbb{D}$ and $\Pi_s^{I_0,\Lambda} = \sum_{i\neq j} f_{ijs}^{I_0,\Lambda}(x^R) Y_i \wedge Y_j$. The computation is the same as in the formal case because for any $1 \leqslant i \leqslant n$ one has the relation $\mathcal{L}_{A_i}(x^R) = 0$, which permits to treat the functions $f_{ijs}^{I_0,\Lambda}(x^R)$ as constants. In this way one shows that if $\mathcal{R}_s^\Lambda$ does not vanish, then the associated vector field is not in normal form. This contradicts theorem (SN).

The main difficulty is to deduce the compatibility relation $[\Pi_2, x^{I_0} \Pi^{I_0,\Lambda}] = 0$ from the Jacobi Identity $[\Phi_s^*(\upsilon_\Lambda \Pi), \Phi_s^*(\upsilon_\Lambda \Pi)] = 0$. In fact we are working *modulo* x^R: there exist pairs of multi-indexes $I', I'' \in \mathbb{D}$ such that x^R divides $x^{I'+I''}$, i.e. $I' + I'' \notin \mathbb{D}$ – as a consequence there is an integer $e \geqslant 1$ such that $x^{I'+I''} f(x^R) = x^{I'+I''-eR} \tilde{f}(x^R)$, where we denote by f, $\tilde{f}$ infinitely flat functions, and $I' + I'' - eR \in \mathbb{D}$.

One shows that for Λ belonging to some subset of $\mathbb{C}^n$ it is possible to deduce the compatibility relation $[\Pi_2, x^{I_0}\Pi^{I_0,\Lambda}] = 0$ from the relation $[\Phi_s^*(v_\Lambda \Pi), \Phi_s^*(v_\Lambda \Pi)] = 0$. One does not work any more on the multi-index $I_0 \in \mathbb{D}$, but instead, on some parameters associated to the functions $f_{ijs}^{I_0,\Lambda}$. In order to obtain these parameters one deduces from the Jacobi Identities

$$\left\{x^\Lambda, \{x_i, x_j\}\right\} + \left\{x_i, \{x_j, x^\Lambda\}\right\} + \left\{x_j \{x^\Lambda, x_i\}\right\} = 0 \quad 1 \leqslant i, j \leqslant n,$$

that the $n(n-1)$ infinitely flat functions $f_{ijs}^{I,\Lambda}$ satisfy some irregular singular linear differential system which depends only on Λ, Π_2 and $x^R \Pi^R$.

3. Application

Following a procedure invented by Martinet and Ramis in [5] one deduces from the sectorial normalizing theorem an analytic classification of Poisson structures as in (P1)–(P7) (cf. [4]).

References

[1] J.P. Dufour, A. Wade, Formes normales de structures de Poisson ayant un 1-jet nul en un point, J. Geom. Phys. 26 (1998) 79–96.

[2] J.P. Dufour, N.T. Zung, Poisson Structures and Their Normal Forms, Progress in Mathematics, vol. 242, Birkhäuser, 2005.

[3] P. Lohrmann, Sur la normalisation holomorphe de structures de Poisson à 1-jet nul, C. R. Acad. Sci. Paris, Ser. I 340 (2005) 823–825.

[4] P. Lohrmann, Classification analytique de structures de Poisson, submitted for publication.

[5] J. Martinet, J.P. Ramis, Problèmes de modules pour les équations differentielles non linéaires de premier ordre, I.H.E.S. 55 (1982) 63–164.

[6] L. Stolovitch, Sur un théorème de Dulac, Ann. Inst. Fourier 44 (5) (1994).

[7] L. Stolovitch, Classification analytique de champs de vecteurs 1-résonnants de $(\mathbb{C}^n, 0)$, Asymptotic Anal. 12 (1996) 91–143.