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Functional Analysis

On the structure of the space of wavelet transforms

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Abstract

Let G be the "$ax+b$"-group with the left invariant Haar measure $\mathrm{d}\nu$ and ψ be a fixed real-valued admissible wavelet on $L_2(\mathbb{R})$. The complete decomposition of $L_2(G, \mathrm{d}\nu)$ onto the space of wavelet transforms $W_\psi(L_2(\mathbb{R}))$ is obtained after identifying the group G with the upper half-plane Π in $\mathbb{C}$. ***To cite this article: O. Hutník, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur l'structure de espace des transformées en ondelette. Soient G le groupe affine « $ax+b$ », $\mathrm{d}\nu$ une mesure de Haar invariante à gauche sur G et ψ une ondelette réelle admissible dans $L_2(\mathbb{R})$. La décomposition complète de $L_2(G, \mathrm{d}\nu)$ sur les espaces des transformées en ondelette $W_\psi(L_2(\mathbb{R}))$ est obtenue, par l'identification du groupe G avec le demi-plan supérieur Π dans $\mathbb{C}$. ***Pour citer cet article : O. Hutník, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let $G = \{\zeta = (u, v);\, u \in \mathbb{R},\, v > 0\}$ be the "$ax+b$"-group with its left invariant Haar measure $\mathrm{d}\nu(\zeta) = v^{-2}\,\mathrm{d}u\,\mathrm{d}v$. The group G acts on $L_2(\mathbb{R})$ via translations and dilations, i.e. for $\zeta = (u, v) \in G$, the unitary representation U_ζ of G is given by $U_\zeta \psi(x) = \psi_{u,v}(x) = v^{-1/2}\,\psi(\frac{x-u}{v})$, where $\psi \in L_2(\mathbb{R})$ is an admissible wavelet, i.e. for almost every $x \in \mathbb{R}$

$$\int_{\mathbb{R}_+} \left|\hat{\psi}(x\xi)\right|^2 \frac{\mathrm{d}\xi}{\xi} = 1, \tag{1}$$

and $\hat{\psi}$ stands for the Fourier transform $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ given by $\mathcal{F}\{g\}(\xi) = \hat{g}(\xi) = \int_{\mathbb{R}} g(x)\mathrm{e}^{-2\pi \mathrm{i} x\xi}\,\mathrm{d}x$. The *Calderón reproducing formula*, cf. [1], is the following resolution of unity on $L_2(\mathbb{R})$,

$$\langle f, g\rangle = \int_G \langle f, \psi_\zeta\rangle\langle\psi_\zeta, g\rangle\,\mathrm{d}\nu(\zeta), \quad (f, g \in L_2(\mathbb{R})),$$

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where the integral is understood in a weak sense. It is not hard to check that the *admissibility condition* (1) is not only sufficient, but also necessary for the Calderón reproducing formula to hold.

In what follows we identify the group G with the upper half-plane $\Pi = \{\zeta = u + \mathrm{i}v;\ u \in \mathbb{R}, v > 0\}$ in the complex plane $\mathbb{C}$. For a fixed admissible wavelet $\psi \in L_2(\mathbb{R})$, the functions $W_\psi f$ (called the *continuous wavelet transforms*) on G of the form

$$W_\psi f(\zeta) = \langle f, \psi_\zeta \rangle, \quad f \in L_2(\mathbb{R}), \tag{2}$$

form a reproducing kernel Hilbert space $W_\psi(L_2(\mathbb{R}))$ called the *space of wavelet* (*or Calderón*) *transforms*. It is a closed subspace of $L_2(G, \mathrm{d}\nu)$. It follows that the transform $W_\psi : L_2(\mathbb{R}) \to L_2(G, \mathrm{d}\nu)$ given by (2) is an isometry, and that the integral operator $P_\psi : L_2(G, \mathrm{d}\nu) \to L_2(G, \mathrm{d}\nu)$ given by

$$P_\psi F(\zeta) = \int_G F(\eta) k(\eta, \zeta)\, \mathrm{d}\nu(\eta), \quad F \in L_2(G, \mathrm{d}\nu),$$

is an orthogonal projection onto $W_\psi(L_2(\mathbb{R}))$, where $k(\eta, \zeta) = \langle \psi_\eta, \psi_\zeta \rangle$ is the reproducing kernel in $W_\psi(L_2(\mathbb{R}))$.

Our aim is to study the associated space of Calderón transforms $W_\psi(L_2(\mathbb{R}))$ and show its structure inside the space $L_2(G, \mathrm{d}\nu)$. The idea of decomposition $L_2(G, \mathrm{d}\nu)$ onto $W_\psi(L_2(\mathbb{R}))$ is based on paper [9]. As motivation for our paper let us mention the following well known result describing the relation between the weighted Bergman spaces on the upper half-plane and the space of Calderón transforms of Hardy space functions with respect to Bergman wavelets, cf. [2,7]:

Let $H_2(\mathbb{R})$ be the Hardy space of all square integrable functions whose Fourier transform is supported on $\mathbb{R}_+$, and take the specific (Bergman) wavelet ψ^α, $\alpha > 0$, defined by

$$\hat{\psi}^\alpha(\xi) = \begin{cases} c_\alpha \xi^\alpha \mathrm{e}^{-2\pi\xi} & \text{for } \xi > 0, \\ 0 & \text{for } \xi \leqslant 0, \end{cases}$$

where

$$c_\alpha = \left(\int_{\mathbb{R}_+} \left| \xi^\alpha \mathrm{e}^{-2\pi\xi} \right|^2 \frac{\mathrm{d}\xi}{\xi} \right)^{-1/2} = \frac{(4\pi)^\alpha}{\sqrt{\Gamma(2\alpha)}}$$

is a normalization factor and $\Gamma(z)$ is the Euler gamma function. Denote by $W_{\psi^\alpha}(H_2(\mathbb{R}))$ the space of wavelet transforms of functions in the Hardy space $H_2(\mathbb{R})$ with respect to the Bergman wavelet ψ^α. Let $A_\beta(G)$, $\beta > -1$, stand for the weighted Bergman space of holomorphic functions on G satisfying

$$\|F\|_{A_\beta}^2 = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left| F(u + \mathrm{i}v) \right|^2 v^\beta \, \mathrm{d}u \, \mathrm{d}v < \infty.$$

Introducing the transform $B^\alpha F(u, v) = v^{-\alpha - 1/2} F(u, v)$ leads to the following:

Theorem 1.1. *The unitary operator B^α gives an isometrical isomorphism of the space $L_2(G, \mathrm{d}\nu)$ onto $L_2(G, v^{2\alpha-1}\, \mathrm{d}u\, \mathrm{d}v)$ under which the space of Calderón transforms $W_{\psi^\alpha}(H_2(\mathbb{R}))$ is mapped onto the weighted Bergman space $A_{2\alpha-1}(G)$.*

The unitary map B^α from W_{ψ^α} to $A_{2\alpha-1}$ provides a unitary equivalence between commutators defined by the wavelet ψ^α and their Bergman space analogues, cf. [6]. Also, this result provides a good tool for studying Toeplitz operators on weighted Bergman spaces on the upper half-plane which (in this case) become unitarily equivalent to Calderón–Toeplitz operators, see [5].

2. Representation of $W_\psi(L_2(\mathbb{R}))$

Our purpose is to give some more general results involving an arbitrary *real-valued admissible wavelet* $\psi \in L_2(\mathbb{R})$ and obtain a new representation of the space $W_\psi(L_2(\mathbb{R}))$.

Introduce the unitary operator $U_1 = (\mathcal{F} \otimes I) : L_2(G, \mathrm{d}\nu(\zeta)) = L_2(\mathbb{R}, \mathrm{d}u) \otimes L_2(\mathbb{R}_+, v^{-2}\,\mathrm{d}v) \to L_2(\mathbb{R}, \mathrm{d}u) \otimes L_2(\mathbb{R}_+, v^{-2}\,\mathrm{d}v)$, with $\zeta = (u, v) \in G$. Let us denote by A_1 the image of the space $W_\psi(L_2(\mathbb{R}))$ by the mapping U_1. The space A_1 consists of all functions

$$F(u,v) = \sqrt{v}\, f(u)\overline{\hat{\psi}(uv)}, \tag{3}$$

where $f \in L_2(\mathbb{R})$ and $\psi \in L_2(\mathbb{R})$ is a real admissible wavelet. Moreover, $\|F(u,v)\|_{A_1} = \|f(u)\|_{L_2(\mathbb{R},\mathrm{d}u)}$. The orthogonal projection $B_1 : L_2(G, \mathrm{d}\nu) \to A_1$ has obviously the form $B_1 = (\mathcal{F} \otimes I)P_\psi(\mathcal{F}^{-1} \otimes I)$, and is given by

$$(B_1F)(u,v) = \sqrt{v}\,\overline{\hat{\psi}(uv)} \int_{\mathbb{R}_+} F(u,t)\hat{\psi}(ut)\frac{\mathrm{d}t}{t^{3/2}}.$$

Thus for each function $F \in L_2(G, \mathrm{d}\nu)$ its image $(B_1F)(u,v)$ has the form (3) with

$$f(u) = \int_{\mathbb{R}_+} F(u,t)\hat{\psi}(ut)\frac{\mathrm{d}t}{t^{3/2}},$$

and therefore, for each function F_0 of the form (3) we have $(B_1F_0)(u,v) = F_0(u,v)$.

Introduce the unitary operator $U_2 : L_2(G, \mathrm{d}\nu) = L_2(\mathbb{R}, \mathrm{d}u) \otimes L_2(\mathbb{R}_+, v^{-2}\,\mathrm{d}v) \to L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y)$ by the formula

$$U_2 : F(u,v) \mapsto U_2F(x,y) = \frac{\sqrt{|x|}}{y} F\left(x, \frac{y}{|x|}\right).$$

Then inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y) \to L_2(\mathbb{R}, \mathrm{d}u) \otimes L_2(\mathbb{R}_+, v^{-2}\,\mathrm{d}v)$ is given by $U_2^{-1} : F(x,y) \mapsto \sqrt{|u|}\, vF(u, |u|v)$. Define $A_2 = U_2(A_1)$. Then the operator $B_2 = U_2B_1U_2^{-1}$ is obviously the orthogonal projection of $L_2(G, \mathrm{d}\nu)$ onto A_2, and

$$(B_2F)(x,y) = \frac{1}{\sqrt{y}}\overline{\hat{\psi}\big(\mathrm{sgn}(x)y\big)} \int_{\mathbb{R}_+} F(x,\tau)\hat{\psi}\big(\mathrm{sgn}(x)\tau\big)\frac{\mathrm{d}\tau}{\sqrt{\tau}}.$$

Now with $F \in A_1$ the space $A_2 \subset L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y)$ consists of all functions of the form

$$(U_2F)(x,y) = \frac{\sqrt{|x|}}{y} F\left(x, \frac{y}{|x|}\right) = \frac{1}{\sqrt{y}}\overline{\hat{\psi}\big(\mathrm{sgn}(x)y\big)} f(x).$$

Introducing $l_\pm(y) = \frac{1}{\sqrt{y}}\overline{\hat{\psi}(\pm y)}$, we have obviously that $l_\pm(y) \in L_2(\mathbb{R}_+, \mathrm{d}y)$ and $\|l_\pm(y)\|_{L_2(\mathbb{R}_+,\mathrm{d}y)} = 1$ (from the admissibility condition). For each $f \in L_2(\mathbb{R})$, let $f_\pm(x) = \chi_\pm(x)f(x)$ be its restriction onto positive and negative half-lines, that is, $\chi_\pm(x)$ are the characteristic functions of $\mathbb{R}_\pm$, respectively. Then

$$(U_2F)(x,y) = f_+(x)l_+(y) + f_-(x)l_-(y).$$

Denote by $L_\pm$ the one-dimensional subspaces of $L_2(\mathbb{R}_+, \mathrm{d}y)$ generated by functions $l_\pm(y)$. Then

$$A_2 = L_2(\mathbb{R}_+) \otimes L_+ \oplus L_2(\mathbb{R}_-) \otimes L_-,$$

and the one-dimensional projections $P_\pm$ of $L_2(\mathbb{R}_+, \mathrm{d}y)$ onto $L_\pm$ have the form

$$(P_\pm H)(y) = \langle H, l_\pm\rangle l_\pm = \frac{1}{\sqrt{y}}\overline{\hat{\psi}(\pm y)} \int_{\mathbb{R}_+} H(\tau)\hat{\psi}(\pm\tau)\frac{\mathrm{d}\tau}{\sqrt{\tau}}. \tag{4}$$

Thus, $B_2 = \chi_+I \otimes P_+ \oplus \chi_-I \otimes P_-$. This leads immediately to the following main result which describes the structure of the space of wavelet transforms $W_\psi(L_2(\mathbb{R}))$ inside $L_2(G, \mathrm{d}\nu)$:

Theorem 2.1. *The unitary operator $U = U_2U_1$ gives an isometrical isomorphism of the space $L_2(G) = L_2(\mathbb{R}, \mathrm{d}u) \otimes L_2(\mathbb{R}_+, v^{-2}\,\mathrm{d}v)$ onto $L_2(\mathbb{R}, \mathrm{d}x) \otimes L_2(\mathbb{R}_+, \mathrm{d}y)$ under which*

(i) *the space of Calderón transforms* $W_\psi(L_2(\mathbb{R}))$ *is mapped onto* $L_2(\mathbb{R}_+) \otimes L_+ \oplus L_2(\mathbb{R}_-) \otimes L_-$ *with*

$$U : W_\psi\big(L_2(\mathbb{R})\big) \mapsto L_2(\mathbb{R}_+) \otimes L_+ \oplus L_2(\mathbb{R}_-) \otimes L_-,$$

where $L_\pm$ *are the one-dimensional subspaces of* $L_2(\mathbb{R}_+, \mathrm{d}y)$ *generated by* $l_\pm(y) = \frac{1}{\sqrt{y}}\overline{\hat{\psi}(\pm y)}$;

(ii) *the projection* P_ψ *is unitarily equivalent to*

$$U P_\psi U^{-1} = \chi_+ I \otimes P_+ \oplus \chi_- I \otimes P_-,$$

where $P_\pm$ *are the one-dimensional projections of* $L_2(\mathbb{R}_+, \mathrm{d}y)$ *onto* $L_\pm$ *given in* (4).

Let us mention that by taking different admissible wavelets ψ_i forming all together an orthonormal basis of L_2 one gets a full decomposition of the space $L_2(G, \mathrm{d}\nu)$. Note that the above representation of the space of Calderón transforms $W_\psi(L_2(\mathbb{R}))$ is especially important in the study of the Calderón–Toeplitz operators with symbols depending only on imaginary part of $\zeta = (u, v) \in G$, cf. [3]. Related operators show up in physics when working with coherent states, cf. [4,8].

References

[1] A. Calderón, Intermediate spaces and interpolation, the complex method, Studia Math. 24 (1964) 113–190.

[2] A. Grossmann, J. Morlet, T. Paul, Transforms associated to square integrable group representations II: Examples, Ann. Inst. Henri Poincaré 45 (1986) 293–309.

[3] O. Hutník, On Toeplitz-type operators related to wavelets, Integral Equations Operator Theory, submitted for publication.

[4] J.R. Klauder, B. Skagerstam, Coherent States, Applications in Physics and Mathematical Physics, World Scientific, Singapore, 1985.

[5] K. Nowak, Weak type estimates for singular values of commutators on weighted Bergman spaces, Indiana Univ. Math. J. 40 (1991) 1315–1331.

[6] K. Nowak, Commutators based on the Calderón reproducing formula, Studia Math. 104 (3) (1993) 285–306.

[7] T. Paul, États quantiques réalisés pas des fonctions analytiques dans le demi-plan, Thèse de 3ième cycle, Université Pierre et Marie Curie, 1984.

[8] T. Paul, Functions analytic on the half-plane as quantum mechanical states, J. Math. Phys. 25 (1984) 3252–3262.

[9] N.L. Vasilevski, On the structure of Bergman and poly-Bergman spaces, Integral Equations Operator Theory 33 (1999) 471–488.