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Numerical Analysis

Some simple error estimates of finite volume approximate solution for parabolic equations

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Abstract

An implicit finite volume scheme for parabolic equations, in which the approximate initial condition is an "orthogonal projection" of the exact initial function, is considered.

In this Note, we prove that the error estimate is of order $h + k$ (where h and k are, respectively, the mesh size of the space discretization and the mesh size of the time discretization) on the discrete norms of $L^\infty(0, T; H_0^1(\Omega))$ and $W^{1,\infty}(0, T; L^2(\Omega))$. From these results, error estimates can be derived for the approximations of the fluxes across the interfaces between neighbouring control volumes and of the first derivative of the unknown solution with respect to the time. ***To cite this article: A. Bradji, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Estimations simples d'erreurs dans la méthode de volumes finis pour des problèmes paraboliques. On considère un schéma implicite de volumes finis pour les problèmes paraboliques. La condition initiale a été discrétisée en utilisant une « projection orthogonale ».

Dans cette Note, on démontre que l'estimation d'erreur est d'ordre $h + k$ (h et k étant, respectivement, le pas de discrétisation en espace et le pas de discrètisation en temps) en normes discrètes dans $L^\infty(0, T; H_0^1(\Omega))$ et dans $W^{1,\infty}(0, T; L^2(\Omega))$. Ces estimations nous permettent d'obtenir des estimations des approximations des flux au travers des interfaces entre les volumes voisins de contrôle et de la dérivée par rapport au temps de la solution inconnue. ***Pour citer cet article : A. Bradji, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

For the sake of simplicity, we consider the following transient diffusion problem:

$$u_t(x, t) - \Delta u(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T), \tag{1}$$

where Ω is an open polygonal bounded domain in $\mathbb{R}^d$, with $d = 2$ or $d = 3$, $T > 0$, and f is a given function.

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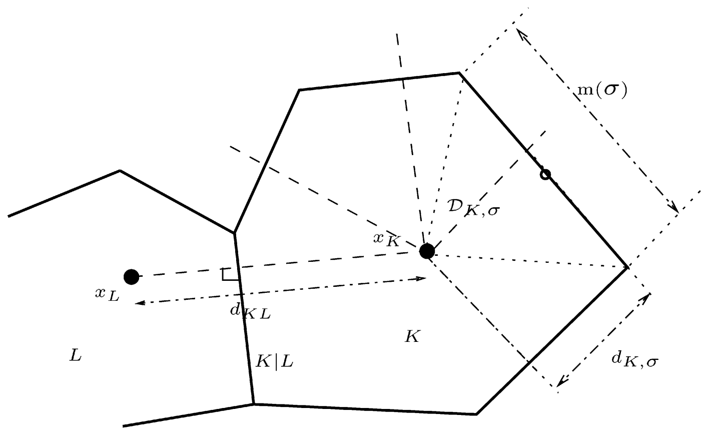

Fig. 1. Notations for a control volume K in the case $d = 2$.

An initial condition is given by:

$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{2}$$

and we consider only the homogeneous Dirichlet boundary conditions, which are:

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{3}$$

To define a finite volume approximation for (1)–(3), we first introduce an admissible mesh $\mathcal{T}$ of Ω in the sense of [2, Definition 9.1 page 762]. An example of two neighbouring control volumes K and L of $\mathcal{T}$ is depicted in Fig. 1. The notations are identical to that of [2].

The size h of the space discretization $\mathcal{T}$ is given by $h = \sup\{\mathrm{diam}(K),\, K \in \mathcal{T}\}$. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$). In the sequel, we use the notation τ_σ to denote $\frac{\mathrm{m}(\sigma)}{d_\sigma}$, and for the sake of simplicity, we assume that $x_K \in K$ for all $K \in \mathcal{T}$. The time discretization is performed with a constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^\star$, and we shall denote $t_n = nk$, for $n \in \{0, \ldots, N+1\}$.

Throughout this paper, the letter C stands for a positive constant independent of the mesh sizes h and k.

Denote by $\{u_K^n;\, K \in \mathcal{T}, n \in \{0, \ldots, N+1\}\}$ the discrete unknowns; the value u_K^n is expected to approximate $u(x_K, nk)$. An implicit finite volume scheme may be given by, see [2, pages 842–843]:

$$\mathrm{m}(K)\frac{u_K^{n+1} - u_K^n}{k} - \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \big(u_L^{n+1} - u_K^{n+1}\big) = \mathrm{m}(K) f_K^n, \quad \forall K \in \mathcal{T}, \forall n \in \{0, \ldots, N\}, \tag{4}$$

where we have denoted $\sigma = K|L$ if $\sigma \in \mathcal{E}_{\mathrm{int}}$, and $u_L^{n+1} = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, and $f_K^n = \frac{1}{k\mathrm{m(K)}} \int_{t_n}^{t_{n+1}} \int_K f(x,t)\,\mathrm{d}x\,\mathrm{d}t$.

We are interested in the following discretization u_K^0 of the initial condition (2), on each control volume K, u_K^0 is defined by:

$$-\sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \big(u_L^0 - u_K^0\big) = -\int_K \Delta u^0(x)\,\mathrm{d}x, \quad \forall K \in \mathcal{T}, \tag{5}$$

where we have denoted, as usual, $\sigma = K|L$ if $\sigma \in \mathcal{E}_{\mathrm{int}}$, and $u_L^0 = 0$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$.

Thanks to the techniques of the proof of [2, Theorem 17.1, page 843], we may prove the existence and uniqueness of a solution $(u_K^n)_{K \in \mathcal{T},\, n \in [\![0, N+1]\!]}$ satisfying (4), (5), and that the following "discrete $L^\infty(0,T; L^2(\Omega))$-estimate" holds, provided that $u \in \mathcal{C}^2(\overline{\Omega} \times [0,T], \mathbb{R})$:

$$\sum_{K \in \mathcal{T}} \mathrm{m}(K)\big(u(x_K, t_n) - u_K^n\big)^2 \leqslant C(h+k)^2, \quad \forall n \in \{0, \ldots, N+1\}.$$

2. Convergence results

The main result of this paper is the following theorem:

Theorem 2.1 *(Error estimate in discrete norms of $L^\infty(0,T;H_0^1(\Omega))$ and $W^{1,\infty}(0,T;L^2(\Omega))$). Let Ω be an open polygonal bounded domain in $\mathbb{R}^d$, $d=2$ or 3, $T>0$ and $u\in\mathcal{C}^2(\overline{\Omega}\times[0,T],\mathbb{R})$. Let $u^0\in\mathcal{C}^2(\overline{\Omega},\mathbb{R})$ be defined by $u^0=u(\cdot,0)$, let $f\in\mathcal{C}(\overline{\Omega}\times[0,T],\mathbb{R})$ be defined by $f=u_t-\Delta u$. Let $\mathcal{T}$ be an admissible mesh of Ω and $k=\frac{T}{N+1}$, with $N\in\mathbb{N}^\star$. Then there exists a unique vector $(u_K^n)_{K\in\mathcal{T},n\in[\![0,N+1]\!]}$ satisfying (4), (5). Let $e_K^n=u(x_K,t_n)-u_K^n$, for $K\in\mathcal{T}$ and $n\in\{0,\dots,N+1\}$. Assume that the exact solution u of (1)–(3) satisfies $u\in\mathcal{C}^4(\overline{\Omega}\times[0,T],\mathbb{R})$. Then the following error estimates hold:*

1. **Error estimate in a discrete semi-norm of** $W^{1,\infty}(0,T;L^2(\Omega))$
$$\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(\frac{e_K^{n+1}-e_K^n}{k}\right)^2\leqslant C(h+k)^2,\quad\forall n\in\{0,\dots,N\},\tag{6}$$
2. **Error estimate in discrete** $L^\infty(0,T;H_0^1(\Omega))$**-norm**
$$\sum_{\sigma\in\mathcal{E}}\tau_\sigma\left(e_L^n-e_K^n\right)^2\leqslant C(h+k)^2,\quad\forall n\in\{0,\dots,N+1\},\tag{7}$$
where we have denoted, as usual, $\sigma=K|L$ if $\sigma\in\mathcal{E}_{\mathrm{int}}$, and $e_L^n=0$ if $\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}_K$.

Sketch of the proof. The main idea of the proof of the estimates (6), (7) is to compare the finite volume solution defined by (4)–(5) with the following finite volume approximation $\{\bar u_K^n\colon K\in\mathcal{T}, n=0,\dots,N+1\}$ defined by (We denote $\sigma=K|L$ if $\sigma\in\mathcal{E}_{\mathrm{int}}$, and $\bar u_L^n=0$ if $\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}_K$.), for all $K\in\mathcal{T}$:
$$-\sum_{\sigma\in\mathcal{E}_K}\tau_\sigma\left(\bar u_L^n-\bar u_K^n\right)=-\int_K\Delta u(x,t_n)\,\mathrm{d}x,\quad\forall n\in\{0,\dots,N+1\}.\tag{8}$$
Let us write $e_K^n=\xi_K^n+\eta_K^n$, with $\xi_K^n=u(x_K,t_n)-\bar u_K^n$ and $\eta_K^n=\bar u_K^n-u_K^n$.
Step 1. (Estimates on $\{\xi_K^n\}$.) Thanks to the error estimate of elliptic equation, see [2, Theorem 9.3, page 781]
$$\sum_{\sigma\in\mathcal{E}}\tau_\sigma\left(\xi_L^n-\xi_K^n\right)^2\leqslant Ch^2\|u\|^2_{\mathcal{C}^2(\overline{\Omega}\times[0,T],\mathbb{R})},\quad\forall n\in\{0,\dots,N+1\}.\tag{9}$$
One remarks that (8) yields, using the notation of the backward difference $\partial v_n=\frac{v_n-v_{n-1}}{k}$,
$$-\sum_{\sigma\in\mathcal{E}_K}\tau_\sigma\left(\partial\bar u_L^n-\partial\bar u_K^n\right)=-\int_K\Delta\big(\partial u(x,t_n)\big)\,\mathrm{d}x,\quad\forall n\in\{1,\dots,N+1\},\tag{10}$$
one can deduce, thanks to the error estimate of elliptic equation, for some C independent of n,
$$\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(\partial\xi_K^n\right)^2\mathrm{d}x\leqslant Ch^2\|u\|^2_{\mathcal{C}^3(\overline{\Omega}\times[0,T],\mathbb{R})},\quad\forall K\in\mathcal{T},\ \forall n\in\{1,\dots,N+1\}.\tag{11}$$
For the same reason stated for (11), we get for some C independent of n (noting that $\partial^2=\partial(\partial)$),
$$\sum_{K\in\mathcal{T}}\mathrm{m}(K)\left(\partial^2\xi_K^{n+1}\right)^2\mathrm{d}x\leqslant Ch^2\|u\|^2_{\mathcal{C}^4(\overline{\Omega}\times[0,T],\mathbb{R})},\quad\forall n\in\{1,\dots,N\}.\tag{12}$$
Step 2. (Estimates on $\{\eta_K^n\}$.) From (8) and (4), we may easily obtain, for all $K\in\mathcal{T}$ and $n\in\{0,\dots,N\}$,
$$\mathrm{m}(K)\partial\eta_K^{n+1}-\sum_{\sigma\in\mathcal{E}_K}\tau_\sigma\left(\eta_L^{n+1}-\eta_K^{n+1}\right)=-\frac{1}{k}\int_K\int_{t_n}^{t_{n+1}}f(x,t)\,\mathrm{d}x\,\mathrm{d}t-\int_K\Delta u(x,t_{n+1})\,\mathrm{d}x+\mathrm{m}(K)\partial\bar u_K^{n+1},\tag{13}$$

which implies in turn, for all $K \in \mathcal{T}$ and $n \in \{1, \ldots, N\}$

$$\mathrm{m}(K)\partial^2\eta_K^{n+1} - \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma\big(\partial\eta_K^{n+1} - \partial\eta_L^{n+1}\big) = -\frac{1}{k}\int_K \partial\left(\int_{t_n}^{t_{n+1}} f(x,t)\,\mathrm{d}t\right)\mathrm{d}x - \int_K \Delta\big(\partial u(x,t_{n+1})\big)\,\mathrm{d}x + \mathrm{m}(K)\partial^2\bar{u}_K^{n+1}. \tag{14}$$

Recall that $\partial^2\eta_K^{n+1} = (\partial\eta_K^{n+1} - \partial\eta_K^n)/k$, substituting f with $u_t - \Delta u$ in (14), using suitable Taylor expansions, (12) and using the techniques of the proof of [2, Theorem 17.1, pages 844–847], (14) yields:

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)\big(\partial\eta_K^{n+1}\big)^2 \leqslant \mathrm{e}^T \sum_{K\in\mathcal{T}} \mathrm{m}(K)\big(\partial\eta_K^1\big)^2 + C(h+k)^2, \quad \forall n \in \{1,\ldots,N\}. \tag{15}$$

Let $n = 0$ in (13), substituting f by $u_t - \Delta u$ in (13), using suitable Taylor expansions, multiplying both sides of resulting equation by $\partial\eta_K^1 = (\eta_K^1 - \eta_K^0)/k$ and using $\eta_K^0 = 0$, re-ordering the sum on the edges of the control volumes, using Cauchy–Schwarz inequality and estimate (11) (for $n = 1$), we get,

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)\big(\partial\eta_K^1\big)^2 \leqslant C(h+k)^2. \tag{16}$$

This, with (15), yields (6). Substituting f by $u_t - \Delta u$ in (13), using suitable Taylor expansions, multiplying (13) by η_K^{n+1}, summing over $K \in \mathcal{T}$, using Cauchy–Schwarz and discrete Poincaré inequalities (see [2, Lemma 9.1, page 765]), we get, thanks to (6),

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma\big(\eta_L^{n+1} - \eta_K^{n+1}\big)^2 \leqslant C(h+k)^2, \quad \forall n \in \{0,\ldots,N\}. \tag{17}$$

This, with inequality (9), (in the step $n+1$) yields (7). (Note that, thanks to (5), (7) holds for $n = 0$.) □

Remark 1 *(An application of Theorem 2.1).* **1.** The estimate (6) means that, since $u \in \mathcal{C}^1(\overline{\Omega} \times [0,T], \mathbb{R})$, $(\frac{u_K^{n+1}-u_K^n}{k})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ approximate the derivatives $(u_t(x_K,t_n))_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ by order $h+k$ in a discrete norm of $L^\infty(0,T;L^2(\Omega))$, i.e., $\sum_{K\in\mathcal{T}} \mathrm{m}(K)(\frac{u_K^{n+1}-u_K^n}{k} - u_t(x_K,t_n))^2 \leqslant C(h+k)^2, \forall n \in \{0,\ldots,N\}$.

2. The estimate (7) means that, since $u \in \mathcal{C}^2(\overline{\Omega} \times [0,T], \mathbb{R})$, $(\frac{u_L^n - u_K^n}{d_\sigma})_{\sigma\in\mathcal{E}, n\in[\![0,N+1]\!]}$ approximate the fluxes across the edges of the control volumes $(\frac{1}{\mathrm{m}(\sigma)}\int_\sigma \nabla u(x,t_n)\cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma(x))_{\sigma\in\mathcal{E}, n\in[\![0,N+1]\!]}$, where $\boldsymbol{n}_{K,\sigma}$ is the normal vector to σ outward to K, by order $h+k$ in a discrete norm of $L^\infty(0,T;L^2(\Omega))$, (see the elliptic case in [2, Theorem 9.3 and Remark 9.10, page 781].

3. It is possible, using results of [1] to provide an approximation, using the finite volume solution $u_\mathcal{T}^k = (u_K^n)_{K\in\mathcal{T}, n\in[\![0,N+1]\!]}$ of (4), (5), for the gradient ∇u of the exact solution u of (1)–(3). We consider the discrete gradient introduced in [1, Definition 2.3, page 333]: $\nabla_\mathcal{T}^k u_\mathcal{T}^k(x,t) = \frac{1}{\mathrm{m}(K)}(\sum_{\sigma\in\mathcal{E}_K} \tau_\sigma(x_\sigma - x_K)(u_L^n - u_K^n))$, where, see [1, (2.3), page 331], $x_\sigma = \frac{1}{\mathrm{m}(\sigma)}\int_\sigma x\,\mathrm{d}\gamma(x)$.

Using triangle inequality, [1, Lemma 2.2, page 333], (7) and [1, Lemma 2.5, page 336], we prove that for each $\theta \in (0, \theta_\mathcal{T}]$, there exists a constant C_θ only depending on θ, d, Ω and u such that $\|\nabla_\mathcal{T}^k u_\mathcal{T}^k - \nabla u\|_{L^2(\Omega)^d} \leqslant C_\theta(h + k)$, $\forall t \in [0,T)$, where (cf. [1, (2.4), page 331]) $\theta_\mathcal{T} = \inf\{\frac{d_{K,\sigma}}{\mathrm{diam}(K)}; K \in \mathcal{T}, \sigma \in \mathcal{E}_K\}$.

4. The regularity assumption $u \in \mathcal{C}^4(\overline{\Omega} \times [0,T], \mathbb{R})$ in Theorem 2.1 could be weakened to u, u_t, u_{tt} in $L^\infty(0,T;H^2(\Omega))$. This last regularity could be reached when Ω is convex, and f and u_0 satisfy some sufficient conditions.

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