Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 346 (2008) 503–507

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Mathematical Analysis

A Billingsley type theorem for Bowen entropy

Ji-Hua Ma [a], Zhi-Ying Wen [b]

[a] *Department of Mathematics, Wuhan University, Wuhan 430072, PR China*
[b] *Department of Mathematics, Tsinghua University, Beijing 100081, PR China*

Received 17 August 2007; accepted after revision 12 March 2008

Available online 11 April 2008

Presented by Jean-Pierre Kahane

Abstract

For subsets of a metric space with a continuous map, Bowen introduced a notion of entropy. In this Note we show that the Bowen entropy can be determined via the local entropies of measures. This result can be considered as an analogue of Billingsley's Theorem for the Hausdorff dimension. ***To cite this article: J.-H. Ma, Z.-Y. Wen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Un théorème de type Billingsley pour l'entropie de Bowen. Pour les sous-ensembles d'un espace métrique muni d'une application continue, Bowen avait introduit une notion d'entropie. Dans cette Note nous démontrons que l'entropie de Bowen peut être déterminée par les entropies locales de mesures. Ce résultat est un analogue du théorème de Billingsley pour la dimension de Hausdorff. ***Pour citer cet article : J.-H. Ma, Z.-Y. Wen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

L'entropie est certainement l'une des caractéristiques essentielles des systèmes dymamiques. Le calcul de différentes entropies est important mais souvent difficile. Cette Note concerne l'estimation de l'entropie de Bowen [2] qui a été introduite pour généralisér les résultats d'entropie topologique classique aux cadres non-compacts. Cette entropie est définie d'une manière similaire à celle de la dimension de Hausdorff. On sait que la dimension de Hausdorff peut être déterminée par le théorème de Billingsley [1]. Dans cette note, nous démontrons pour l'entropie de Bowen un résultat analogue au théorème de Billingsley.

Nous commençons par des définitions (voir [6, p. 12]). Soit (X, d) un espace métrique et $f : X \to X$ une application continue. Pour tout entier $n \geqslant 1$, tout nombre réel $r > 0$, et tout $x \in X$, la (n, r)-boule centrée en x est définie par

$$B_n(x, r) = \big\{ y \in X\colon\ d(f^i x, f^i y) \leqslant r \text{ pour } 0 \leqslant i \leqslant n-1 \big\}.$$

E-mail addresses: jhma@whu.edu.cn (J.-H. Ma), wenzy@tsinghua.edu.cn (Z.-Y. Wen).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.03.010

Fixons $r > 0$. Pour $E \subset X$, $s \geqslant 0$, et $N \geqslant 1$, posons

$$\mathcal{H}_N^s(E,r) = \inf\Big\{\sum_{i\geqslant 1} \mathrm{e}^{-sn_i}\colon \{B_{n_i}(x_i,r)\}_{i\geqslant 1} \text{ est un } (N,r)\text{-recouvrement de } E\Big\},$$

où une collection finie ou dénombrable de boules $\{B_{n_i}(x_i,r)\}_{i\geqslant 1}$ est dite un (N,r)-recouvrement de E, si $E \subset \bigcup_{i\geqslant 1} B_{n_i}(x_i,r)$, et $n_i \geqslant N$ pour tout $i \geqslant 1$. Lorsque N croît, la classe de (N,r)-recouvrements de E diminue, et donc la borne inférieure croît. Ainsi la limite $\mathcal{H}^s(E,r) = \lim_{N\to\infty} \mathcal{H}_N^s(E,r)$ existe, et définit une mesure métrique extérieure de sous-ensembles de X. Voir [6, p.12] pour plus de détails.

Définition 1. Pour tout $r > 0$ fixé et tout $E \subset X$, la r-entropie topologique de E est définie par

$$h(E,r) = \sup\{s\colon \mathcal{H}^s(E,r) > 0\} = \inf\{s\colon \mathcal{H}^s(E,r) = 0\}.$$

L'entropie de Bowen de E (par rapport à f) est définie comme $h(E) = \lim_{r\to 0} h(E,r)$.

Notre résultat principal est un analogue au théorème de Billingsley pour la dimension de Hausdorff (voir [1, p. 141, Theorem 14.1]). Plus précisément, nous allons établir une estimation fine de l'entropie de Bowen de sous-ensembles de X via l'entropie locale de certaines mesures chargeant ces sous-ensembles.

Définition 2. Soit μ une mesure borélienne de probabilité sur X. L'entropie locale inférieure de μ est définie par

$$h_\mu(x) = \lim_{r\to 0} \liminf_{n\to\infty} \frac{-\log \mu(B_n(x,r))}{n}.$$

L'entropie locale supérieure peut être définie d'une façon similaire. Quand l'entropie locale supérieure et l'entropie locale inférieure coïncident, la valeur commune sera appelée entropie locale. L'entropie locale d'une mesure invariante est liée au théorème de Shannon–MacMillan–Breiman (voir [3]).

Théorème 1. *Soit μ une mesure borélienne de probabilité sur X, E un sous-ensemble borélien de X, et* $0 < s < \infty$.

(1) *Si $h_\mu(x) \leqslant s$ pour tout $x \in E$, alors $h(E) \leqslant s$.*
(2) *Si $h_\mu(x) \geqslant s$ pour tout $x \in E$ et $\mu(E) > 0$, alors $h(E) \geqslant s$.*

Comme application, on obtient une autre preuve simple du résultat de Bowen suivant [2, p. 128–130].

Proposition 1. *Soit (X,d) un espace métrique compact, $f: X \to X$ une application continue, et μ une mesure borélienne f-invariante de probabilité sur X. Si $E \subset X$ et $\mu(E) = 1$, alors $h(E) \geqslant h_\mu$, où h_μ est l'entropie de mesure classique de μ.*

1. Definitions and results

Entropy is undoubtedly among the most essential characteristics of dynamical systems. Calculations of various entropies are important but often difficult. We are concerned with the estimation of Bowen entropy [2], which was introduced to extend results about the classical topological entropy to non-compact settings. This entropy is defined in a way similar to that of Hausdorff dimension. Often the Hausdorff dimension can be determined with the help of Billingsley's theorem [1, p. 141, Theorem 14.1]. For the Bowen entropy, we show in this note that an analogue of the Billingsley's Theorem does exist.

We begin with some definitions (see [6, p. 12]). Let f be a continuous map on a metric space (X,d). For any integer $n \geqslant 1$, real number $r > 0$ and $x \in X$, define the (n,r)-ball centered at x by

$$B_n(x,r) = \{y \in X\colon d(f^i x, f^i y) \leqslant r \text{ for } 0 \leqslant i \leqslant n-1\}.$$

Fix a real number $r > 0$. For any $E \subset X$, $s \geqslant 0$ and $N \geqslant 1$, define

$$\mathcal{H}_N^s(E,r) = \inf\Big\{\sum_{i\geqslant 1} \mathrm{e}^{-sn_i}\colon \{B_{n_i}(x_i,r)\}_{i\geqslant 1} \text{ is an } (N,r)\text{-cover of } E\Big\},$$

where, by an (N,r)-**cover** of E, we mean a finite or countable collection of balls $\{B_{n_i}(x_i,r)\}_{i\geqslant 1}$ such that $E\subset\bigcup_{i\geqslant 1}B_{n_i}(x_i,r)$ and $n_i\geqslant N$ for all i. By convention, we put $\mathcal{H}_N^s(E,r)=\infty$ if no (N,r)-**cover** of E exists. Observe that as N increases the class of (N,r)-covers of E is reduced and the infimum increases. Therefore, the limit $\mathcal{H}^s(E,r)=\lim_{N\to\infty}\mathcal{H}_N^s(E,r)$ exists and determines a metric outer measure on subsets of X (see [6] for more details).

Definition 1. For any $r>0$ and $E\subset X$, the r-topological entropy of E is defined as

$$h(E,r)=\sup\bigl\{s\colon\ \mathcal{H}^s(E,r)>0\bigr\}=\inf\bigl\{s\colon\ \mathcal{H}^s(E,r)=0\bigr\}.$$

The Bowen entropy of E (with respect to f) is defined as $h(E)=\lim_{r\to 0}h(E,r)$.

The Bowen entropy shares some basic properties of the Hausdorff dimension. In particular, we have $h(E_1)\leqslant h(E_2)$ if $E_1\subset E_2$ (the monotonicity), and $h(\bigcup_{n=1}^{\infty}E_n)=\sup\{h(E_n):n\geqslant 1\}$ (the countable stability). Besides, $h(X)$ equals the classical topological entropy when X is compact.

The main purpose of this Note is to show that the Bowen entropy of subsets of X can be estimated via local entropy of certain measures supported on them.

Definition 2. Let μ be a Borel probability measure on X. Then we call

$$h_\mu(x)=\lim_{r\to 0}\liminf_{n\to\infty}\frac{-\log\mu(B_n(x,r))}{n}$$

the lower local entropy of μ at the point $x\in X$.

Remark 1. One can define the upper local entropy in a similar way. The local entropy for invariant measures was related to the Shannon–MacMillan–Breiman theorem in [3].

Theorem 1. *Let μ be a Borel probability measure on X, E be a Borel subset of X and* $0<s<\infty$.

(1) *If $h_\mu(x)\leqslant s$ for all $x\in E$, then $h(E)\leqslant s$.*
(2) *If $h_\mu(x)\geqslant s$ for all $x\in E$ and $\mu(E)>0$, then $h(E)\geqslant s$.*

As application, we give below an alternative proof of the following result of Bowen [2, p. 128–130] which generalizes one-half of the well-known variational principle for entropy:

Proposition 2. *Let X be a compact metric space and $f:X\to X$ a continuous map. Let μ be an f-invariant Borel probability measure with measure-theoretic entropy h_μ. Then we have $h_\mu\leqslant h(E)$ for all $E\subset X$ with $\mu(E)=1$.*

Proof. By the Brin and Katok's Theorem [3], $\int h_\mu(x)\,\mathrm{d}\mu(x)=h_\mu$. It follows that the set

$$E_\delta:=\bigl\{x\in E\colon\ h_\mu(x)\geqslant h_\mu-\delta\bigr\}$$

has positive μ-measure for all $\delta>0$ by virtue of $\mu(E)=1$. Thus $h(E_\delta)\geqslant h_\mu-\delta$ by (2) of Theorem 1. Since $E_\delta\subset E$ for all $\delta>0$, one has $h(E)\geqslant h_\mu$ as desired. □

Remark 2. In practical application, Theorem 1 is most useful in bounding the Bowen entropy from below. Like in the application of the Billingsley's Theorem to the Hausdorff dimension, often one can choose a subset which carries a simple-structured measure, and then Theorem 1 applies. Let us mention that, this method can be utilized to simplify the proof of the lower bound results for the Bowen entropy of the saturated sets in [7].

The next section is devoted to the proof of Theorem 1.

2. Proof of the main results

First, we prove a lemma which is much like the classical covering lemma (see [4, 2.8.4]):

Lemma 1. *Let $r > 0$ and $\mathcal{B}(r) = \{B_n(x,r)\colon\ x \in X,\ n = 1,2,\ldots\}$. For any family $\mathcal{F} \subset \mathcal{B}(r)$, there exists a (not necessarily countable) subfamily $\mathcal{G} \subset \mathcal{F}$ consisting of disjoint balls such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B_n(x,r)\in\mathcal{G}} B_n(x,3r).$$

Proof. Let Ω denote the partially ordered (by inclusion) set consisting of all subfamilies ω of $\mathcal{F}$ with the following property: (a) ω consists of disjoint balls from $\mathcal{F}$; (b) If a ball $B_n(x,r) \in \mathcal{F}$ meets some ball from ω, then there exists $B_m(y,r) \in \omega$ such that $m \leqslant n$ and $B_n(x,r) \cap B_m(y,r) \neq \emptyset$.

Ω is non-empty. Indeed, the family consisting of a ball $B_m(y,r) \in \mathcal{F}$ with $m = \min\{n\colon\ B_n(x,r) \in \mathcal{F}\}$ is in Ω. Then, let $\mathcal{C} \subset \Omega$ be a chain (viz. a linearly ordered subset), then $\bigcup_{\omega\in\mathcal{C}} \omega$ belongs to Ω and is an upper bound of $\mathcal{C}$. By Zorn's lemma there exists a maximal element $\mathcal{G}$ in Ω.

We claim that each ball in $\mathcal{F}$ does meet some ball in $\mathcal{G}$, and the conclusion follows by the triangle inequality. If this were not true, then we could find a ball $B_m(y,r) \in \mathcal{F}$ such that

$$m = \min\bigl\{n\colon\ B_n(x,r) \text{ does not meet any ball in } \mathcal{G}\bigr\}.$$

It is easy to verify that the family $\mathcal{G} \cup \{B_m(y,r)\}$ satisfies (a) and (b) and hence belongs to Ω. This contradicts the maximality of $\mathcal{G}$. □

Now we proceed with the proof of Theorem 1.

(1) For a fixed $\epsilon > 0$, since $h_\mu(x) \leqslant s$ for all $x \in E$, we have $E = \bigcup_{k=1}^{\infty} E_k$, where

$$E_k = \left\{x \in E\colon\ \liminf_{n\to\infty} \frac{-\log\mu(B_n(x,r))}{n} < s + \epsilon \text{ for all } r \in \left(0, \frac{1}{k}\right)\right\}.$$

Now fix $k \geqslant 1$ and $0 < r < \frac{1}{3k}$. For each $x \in E_k$, there exists a strictly increasing sequence $\{n_j(x)\}_{j=1}^{\infty}$ such that $\mu(B_{n_j(x)}(x,r)) \geqslant e^{-(s+\epsilon)n_j(x)}$ for all $j \geqslant 1$. So, for any $N \geqslant 1$, the set E_k is contained in the union of the sets in the family $\mathcal{F} = \{B_{n_j(x)}(x,r)\colon\ x \in E_k, n_j(x) \geqslant N\}$. By Lemma 1, there exists a sub family $\mathcal{G} = \{B_{n_i}(x_i,r)\}_{i\in I} \subset \mathcal{F}$ consisting of disjoint balls such that

$$E_k \subset \bigcup_{i\in I} B_{n_i}(x_i,3r) \quad \text{and} \quad \mu\bigl(B_{n_i}(x_i,r)\bigr) \geqslant e^{-(s+\epsilon)n_i} \quad \text{for all } i \in I. \tag{1}$$

The index set I is at most countable since μ is a probability measure and $\mathcal{G}$ is a disjointed family of sets, each of which has positive μ-measure. Therefore, $\{B_{n_i}(x_i,3r)\}_{i\in I}$ is an $(N,3r)$-covering of E_k, and consequently

$$\mathcal{H}_N^{s+\epsilon}(E_k,3r) \leqslant \sum_{i\in I} e^{-(s+\epsilon)n_i} \leqslant \sum_{i\in I} \mu\bigl(B_{n_i}(x_i,r)\bigr) \leqslant 1,$$

where the disjointness of $\{B_{n_i}(x_i,r)\}_{i\in I}$ is used in the last inequality. It follows that

$$\mathcal{H}^{s+\epsilon}(E_k,3r) = \lim_{N\to\infty} \mathcal{H}_N^{s+\epsilon}(E_k,3r) \leqslant 1,$$

which, in turn, implies that $h(E_k,3r) \leqslant s + \epsilon$ for any $0 < r < \frac{1}{3k}$. Letting $r \to 0$ yields

$$h(E_k) \leqslant s + \epsilon \quad \text{for any } k \geqslant 1.$$

Since the Bowen entropy is countably stable (see [2, p. 127]), it follows that

$$h(E) = h\left(\bigcup_{k=1}^{\infty} E_k\right) = \sup_{k\geqslant 1}\bigl\{h(E_k)\bigr\} \leqslant s + \epsilon.$$

Therefore, $h(E) \leqslant s$ since $\epsilon > 0$ is arbitrary.

(2) Let us first fix an $\epsilon > 0$. For each $k \geqslant 1$, put

$$E_k = \left\{ x \in E\colon \liminf_{n\to\infty} \frac{-\log \mu(B_n(x,r))}{n} > s - \epsilon \quad \text{for all } r \in \left(0, \frac{1}{k}\right) \right\}.$$

Since $h_\mu(x) \geqslant s$ for all $x \in E$, the sequence $\{E_k\}_{k=1}^\infty$ increases to E. So by the continuity of the measure (see [5] p. 9), we have

$$\lim_{k\to\infty} \mu(E_k) = \mu(E) > 0.$$

Then fix some $k \geqslant 1$ with $\mu(E_k) > \frac{1}{2}\mu(E)$. For each $N \geqslant 1$, put

$$E_{k,N} = \left\{ x \in E_k\colon \frac{-\log \mu(B_n(x,r))}{n} > s - \epsilon \quad \text{for all } n \geqslant N \text{ and } r \in \left(0, \frac{1}{k}\right) \right\}.$$

Since the sequence $\{E_{k,N}\}_{N=1}^\infty$ increases to E_k, we may pick an $N^* \geqslant 1$ such that $\mu(E_{k,N^*}) > \frac{1}{2}\mu(E_k)$. Write $E^* = E_{k,N^*}$ and $r^* = \frac{1}{k}$. Then $\mu(E^*) > 0$ and

$$\mu\big(B_n(x,r)\big) \leqslant \mathrm{e}^{-(s-\epsilon)n} \quad \text{for all } x \in E^*, 0 < r \leqslant r^* \text{ and } n \geqslant N^*. \tag{2}$$

Now suppose that $\mathcal{F} = \{B_{n_i}(y_i, \frac{r}{2})\}_{i\geqslant 1}$ is an $(N, \frac{r}{2})$-cover of E^* such that

$$E^* \cap B_{n_i}\left(y_i, \frac{r}{2}\right) \neq \emptyset, \quad n_i \geqslant N \geqslant N^* \quad \text{for all } i \geqslant 1 \text{ and } 0 < r \leqslant r^*.$$

For each $i \geqslant 1$, there exists an $x_i \in E^* \cap B_{n_i}(y_i, \frac{r}{2})$. By the triangle inequality

$$B_{n_i}\left(y_i, \frac{r}{2}\right) \subset B_{n_i}(x_i, r).$$

In combination with (2), this implies

$$\sum_{i\geqslant 1} \mathrm{e}^{-(s-\epsilon)n_i} \geqslant \sum_{i\geqslant 1} \mu\big(B_{n_i}(x_i, r)\big) \geqslant \mu(E^*).$$

Therefore, $\mathcal{H}_N^{s-\epsilon}(E^*, \frac{r}{2}) \geqslant \mu(E^*) > 0$ for all $N > N^*$, and consequently

$$\mathcal{H}^{s-\epsilon}\left(E^*, \frac{r}{2}\right) = \lim_{N\to\infty} \mathcal{H}_N^{s-\epsilon}\left(E^*, \frac{r}{2}\right) \geqslant \mu(E^*) > 0,$$

which in turn implies that $h(E^*, \frac{r}{2}) \geqslant s - \epsilon$. Then we have $h(E^*) \geqslant s - \epsilon$ by letting $r \to 0$. It follows that $h(E) \geqslant h(E^*) \geqslant s - \epsilon$, and hence $h(E) \geqslant s$ since $\epsilon > 0$ is arbitrary. The proof is completed now. □

Acknowledgements

The authors are supported by the Chinese National Science Fund No. 10571104 and No. 10771164.

References

[1] P. Billingsley, Ergodic Theory and Information, John Wiley and Sons Inc., New York, 1965.
[2] R. Bowen, Topological entropy for noncompact sets, Trans. Amer. Math. Soc. 184 (1973) 125–136.
[3] M. Brin, A. Katok, On local entropy, in: Geometric Dynamics, in: Lecture Notes in Mathematics, vol. 1007, Springer, Berlin, 1983, pp. 30–38.
[4] H. Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.
[5] P. Mattilla, Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, Cambridge, 1995.
[6] Y. Pesin, Dimension Theory in Dynamical Systems: Contemporary Views and Applications, The University of Chicago Press, Chicago and London, 1997.
[7] C.E. Pfister, W.G. Sullivan, On the topological entropy of saturated sets, Ergodic Theory Dynam. Systems 27 (2007) 929–956.