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Mathematical Problems in Mechanics

Global weak solutions for asymmetric incompressible fluids with variable density

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Abstract

We establish the existence of global in time weak solutions for the equations of asymmetric incompressible fluids with variable density, when the initial density is not necessarily strictly positive. ***To cite this article: P. Braz e Silva, E.G. Santos, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Solutions faibles globales pour les équations des fluides incompressibles asymétriques à densité variable. On établit l'existence de solutions faibles globales en temps pour les équations des fluides incompressibles asymétriques à densité variable, dans le cas oú la densité initiale n'est pas strictement positive. ***Pour citer cet article : P. Braz e Silva, E.G. Santos, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set. We are interested in the flow of an asymmetric incompressible fluid with variable density in Ω. So, for a given time $T > 0$, we consider the equations

$$\begin{cases} \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mu+\mu_r)\Delta\mathbf{u} + \nabla p = 2\mu_r \operatorname{curl}\mathbf{w} + \rho\mathbf{f}, \\ \rho(\mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w}) - (c_a+c_d)\Delta\mathbf{w} - (c_0+c_d-c_a)\nabla(\operatorname{div}\mathbf{w}) + 4\mu_r\mathbf{w} = 2\mu_r \operatorname{curl}\mathbf{u} + \rho\mathbf{g}, \\ \operatorname{div}\mathbf{u} = 0, \quad \rho_t + \mathbf{u}\cdot\nabla\rho = 0, \end{cases} \tag{1}$$

in $\Omega \times (0, T)$, with initial and boundary conditions

$$\mathbf{u}(x,t) = \mathbf{w}(x,t) = 0, \quad \forall (x,t) \in \partial\Omega \times (0,T), \tag{2}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{w}(x,0) = \mathbf{w}_0(x), \quad \rho(x,0) = \rho_0(x), \quad \forall x \in \Omega. \tag{3}$$

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The unknowns $\mathbf{u}$, $\mathbf{w}$, ρ, and p are, respectively, the linear velocity, the angular velocity of rotation of fluid particles, the mass density and the pressure distribution of the fluid. The functions $\mathbf{f}$ and $\mathbf{g}$ are given external forces. The positive constants μ, μ_r, c_0, c_a, c_d are related with viscosity properties of the fluid, and satisfy $c_0 + c_d > c_a$.

For the derivation of Eqs. (1) and a discussion about their physical meaning, see [3]. Concerning applications, the micropolar fluid model has been used, for example, in lubrication theory [4,7], as well as in modelling blood flow in thin vessels [1].

In [2], some existence and uniqueness results for strong solutions are given, in the case of a strictly positive initial density. In [5], local in time existence of weak solutions was established (see also [6]). For this local result though, the initial density is required to satisfy the integrability condition $\|\rho_0^{-1}\|_{L^3} < \infty$. Here, we announce the existence of global in time weak solutions, requiring the initial density to be only nonnegative, that is, $\rho_0 \geqslant 0$ (see Theorem 2.1).

Our results bring the knowledge about weak solutions of system (1) to the same level of the knowledge about weak solutions of the variable density Navier–Stokes system [8,9].

2. Preliminaries

We denote by $\mathcal{D}(\Omega)$ the space of test functions defined in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions over Ω. We use the usual notation for Sobolev spaces

$$W^{m,q}(\Omega) = \left\{ f \in L^q(\Omega);\, \|D^\alpha f\|_{L^q(\Omega)} < +\infty,\, |\alpha| \leqslant m \right\},$$

for a multi-index α, a nonnegative integer m and $1 \leqslant q \leqslant +\infty$. We write $H^m(\Omega) := W^{m,2}(\Omega)$ and denote by $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$. If B is a Banach space and $T > 0$, we denote by $L^q([0,T];B)$ the Banach space of B valued functions defined on the interval $[0,T]$ that are L^q-integrable in Bochner's sense. By $\mathcal{D}(0,T;B)$, we denote the space of B valued C^∞ functions defined on $[0,T]$, with compact suppport in $(0,T)$. Accordingly, we indicate the space of distributions with values in B by $\mathcal{D}'(0,T;B)$. As it is usual in this context, we denote $\mathbb{R}^3$ valued functions by bold face letters. We write

$$\mathcal{V} = \left\{ \mathbf{v} \in \left(\mathcal{D}(\Omega)\right)^3;\, \nabla \cdot \mathbf{v} = 0 \right\},$$

and denote by H and V the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ and $(H_0^1(\Omega))^3$ respectively.

Some spaces which are not so standard but play an important role in our results are the Nikolskii spaces, defined as follows: Let B be a Banach space. Given a function $f : (0,T) \to B$ and $h > 0$, let $\tau_h f : (-h, T-h) \to B$ be the translated function of f, defined by $(\tau_h f)(t) = f(t+h)$. For $1 \leqslant q \leqslant \infty$, $0 < s < 1$, the Nikolskii space $N^{s,q}$ is defined by

$$N^{s,q}(0,T;B) := \left\{ f \in L^q(0,T;B);\, \sup_{h>0} h^{-s} \|\tau_h f - f\|_{L^q(0,T-h;B)} < \infty \right\}.$$

The space $N^{s,q}(0,T;B)$ is a Banach space with respect to the norm

$$\|f\|_{N^{s,q}(0,T;B)} := \|f\|_{L^q(0,T;B)} + \sup_{0<h<T} \left[h^{-s} \|\tau_h f - f\|_{L^q(0,T-h;B)} \right].$$

One may see [9–11] for compactness properties of Nikolskii spaces. Our result is the following:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with Lipschitz boundary. Given $T > 0$, if $\mathbf{u}_0 \in H$, $\mathbf{w}_0 \in (L^2(\Omega))^3$, $\rho_0 \in L^\infty(\Omega)$, $\rho_0 \geqslant 0$, and $\mathbf{f}, \mathbf{g} \in L^1(0,T;(L^2(\Omega))^3)$, then there exist*

$$\mathbf{u} \in L^2(0,T;V), \quad \mathbf{w} \in L^2\left(0,T;\left(H_0^1(\Omega)\right)^3\right), \quad p \in W^{-1,\infty}\left(0,T;L^2(\Omega)\right), \quad \rho \in L^\infty\left(0,T;L^\infty(\Omega)\right),$$

such that

$$\rho\mathbf{u},\ \rho\mathbf{w} \in L^\infty\left(0,T;\left(L^2(\Omega)\right)^3\right) \cap N^{\frac{1}{4},2}\left(0,T;\left(W^{-1,3}(\Omega)\right)^3\right),$$
$$\inf_\Omega \rho_0 \leqslant \rho(x,t) \leqslant \sup_\Omega \rho_0,$$

satisfying the equations

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div}(\rho \mathbf{u}\mathbf{u}) - (\mu + \mu_r)\Delta \mathbf{u} + \nabla p = 2\mu_r \operatorname{curl} \mathbf{w} + \rho \mathbf{f},$$
$$\frac{\partial \rho \mathbf{w}}{\partial t} + \operatorname{div}(\rho \mathbf{u}\mathbf{w}) - (c_a + c_d)\Delta \mathbf{w} - (c_0 + c_d - c_a)\nabla \operatorname{div} \mathbf{w} + 4\mu_r \mathbf{w} = 2\mu_r \operatorname{curl} \mathbf{u} + \rho \mathbf{g},$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \operatorname{div} \mathbf{u} = 0,$$

in $\Omega \times (0, T)$*, the boundary conditions* (2)*, and the weak initial conditions*

$$\rho|_{t=0} = \rho_0,$$
$$\left(\int_\Omega \rho \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}x\right)(0) = \int_\Omega \rho_0 \mathbf{u}_0 \cdot \mathbf{v}, \quad \forall \mathbf{v} \in V,$$
$$\left(\int_\Omega \rho \mathbf{w} \cdot \mathbf{z}\, \mathrm{d}x\right)(0) = \int_\Omega \rho_0 \mathbf{w}_0 \cdot \mathbf{z}, \quad \forall \mathbf{z} \in \left(H_0^1(\Omega)\right)^3.$$

Remark 1. In [9], the author obtains $\rho \mathbf{u} \in N^{\frac{1}{4},2}(0, T; (W^{-1,\frac{3}{2}}(\Omega))^3)$. The theorem above assures improved regularity for $\rho \mathbf{u}$, $\rho \mathbf{w}$, and its counterpart for the variable density Navier–Stokes was actually established in [8].

Sketch of the proof. The proof of Theorem 2.1 is based on a semi-Galerkin method, adapting the techniques used in [9] for the variable density Navier–Stokes system to treat our case of asymmetric fluids. First, one defines V^m, W^m, suitable finite dimensional subspaces of V and $Ł^2(\Omega)^3$, respectively. Then, choose sequences $\mathbf{f}^m$, $\mathbf{g}^m \in C([0, T]; (L^2(\Omega))^3)$, $\mathbf{u}_0^m \in V^m$, $\mathbf{w}_0^m \in W^m$, and $\rho_0^m \in C^1(\overline{\Omega})$ such that

$$\frac{1}{m} + \inf_\Omega \rho_0 \leqslant \rho_0^m \leqslant \frac{1}{m} + \sup_\Omega \rho_0 \quad \text{in } \Omega, \text{ for all } m = 1, 2, \ldots,$$

and $\mathbf{f}^m \to \mathbf{f}$, $\mathbf{g}^m \to \mathbf{g}$ in $L^1(0, T; (L^2(\Omega))^3)$, $\mathbf{u}_0^m \to \mathbf{u}_0$ in H, $\mathbf{w}_0^m \to \mathbf{w}_0$ in $(L^2(\Omega))^3$, $\rho_0^m \to \rho_0$ weak-$\star$ in $L^\infty(\Omega)$. Now, for $m \in \mathbb{N}$, we call the triplet $(\rho^m, \mathbf{u}^m, \mathbf{w}^m)$ an (mth) approximate solution of problem (1)–(3) if $\rho^m \in C^1(\overline{\Omega})$, $\mathbf{u}^m \in C^1([0, T]; V^m)$, $\mathbf{w}^m \in C^1([0, T]; W^m)$ satisfy the equations

$$\int_\Omega \left(\rho^m\left(\mathbf{u}_t^m + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^m - \mathbf{f}^m\right) \cdot \mathbf{v} + (\mu + \mu_r)\nabla \mathbf{u}^m \cdot \nabla \mathbf{v} - 2\mu_r \mathbf{w}^m \cdot \operatorname{curl} \mathbf{v}\right) \mathrm{d}x = 0, \quad \forall \mathbf{v} \in V^m, \tag{4}$$

$$\int_\Omega \left(\rho^m\left(\mathbf{w}_t^m + (\mathbf{u}^m \cdot \nabla)\mathbf{w}^m - \mathbf{g}^m\right) \cdot \mathbf{z} + (c_a + c_d)\nabla \mathbf{w}^m \cdot \nabla \mathbf{z} + (c_0 + c_d - c_a) \operatorname{div} \mathbf{w}^m \cdot \operatorname{div} \mathbf{z} + 4\mu_r \mathbf{w}^m \cdot \mathbf{z} - 2\mu_r \mathbf{u}^m \cdot \operatorname{curl} \mathbf{z}\right) \mathrm{d}x = 0, \quad \forall \mathbf{z} \in W^m, \tag{5}$$

$$\rho_t^m + \mathbf{u}^m \cdot \nabla \rho^m = 0, \tag{6}$$

and the initial conditions $\mathbf{u}^m|_{t=0} = \mathbf{u}_0^m$, $\mathbf{w}^m|_{t=0} = \mathbf{w}_0^m$, and $\rho^m|_{t=0} = \rho_0^m$ in Ω. After establishing the existence of approximate solutions, one derives a priori bounds for them. Since one does not have a positive lower bound for the initial density, this task is a little bit harder than usual. The key idea here is to derive bounds for the products $\rho \mathbf{u}$, $\rho \mathbf{w}$. In special, after considerable work one obtains

$$\left\|\tau_h(\rho^m \mathbf{u}^m) - \rho^m \mathbf{u}^m\right\|_{L^2(0,T-h;(W^{-1,3}(\Omega))^3)} \leqslant C h^{\frac{1}{4}},$$
$$\left\|\tau_h(\rho^m \mathbf{w}^m) - \rho^m \mathbf{w}^m\right\|_{L^2(0,T-h;(W^{-1,3}(\Omega))^3)} \leqslant C h^{\frac{1}{4}},$$

where C is independent of m, which imply $\rho \mathbf{u}$, $\rho \mathbf{w} \in N^{\frac{1}{4},2}(0, T; (W^{-1,3}(\Omega))^3)$. These bounds, together with some other ones, allow one to pass to the limit $m \to \infty$ obtaining the desired solution. □

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