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Probability Theory

Convergence of the normalized maximum of regularly varying random functions in the space $\mathbb{D}$

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Abstract

Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. random functions in the space $\mathbb{D}$ of cadlag functions. The purpose of this note is to complement the result of de Haan and Lin (2001) on the link between regular variation of ξ and convergence of the normalized maximum $n^{-1}\bigvee_{i=1}^{n}\xi_i$ in the space $\mathbb{C}$ of continuous functions. We study when regular variation implies convergence of the normalized maximum in $\mathbb{D}$. After exhibiting an example, which shows that this is not true in the general case, we give a sufficient condition under which this implication takes place. ***To cite this article: Y. Gentric, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Convergence du maximum renormalisé de fonctions aléatoires dans l'espace $\mathbb{D}$. Soit $\xi, \xi_1, \xi_2, \ldots$ des fonctions aléatoires i.i.d. dans l'espace $\mathbb{D}$ des fonctions cadlag. De Hann et Lin (2001) ont étudié le lien entre la variation régulière de ξ et la convergence en loi dans $\mathbb{C}$ du maximum renormalisé $n^{-1}\bigvee_{i=1}^{n}\xi_i$. Après avoir exhibé un contre-exemple qui montre que le résultat est faux en toute généralité dans $\mathbb{D}$, nous donnons une condition suffisante qui assure la convergence du maximum renormalisé dans $\mathbb{D}$. A titre d'exemple, le cas d'un processus de Lévy est traité. ***Pour citer cet article : Y. Gentric, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Soit $\xi, \xi_1, \xi_2, \ldots$ des vecteurs ou des fonctions aléatoires i.i.d. Le lien entre la variation régulière de ξ et la convergence en loi du maximum normalisé $n^{-1}\bigvee_{i=1}^{n}\xi_i$ a été étudié dans le cas où ξ est un vecteur aléatoire de dimension finie (cf. [5]). Plus récemment, de Haan et Lin ont généralisé cette notion au cas où ξ est un processus stochastique (cf. [3]). Si on se place dans l'espace $\mathbb{C}$ des fonctions continues de [0, 1] dans $\mathbb{R}$, on a le même résultat qu'en dimension finie (cf. [3], Théorème 2.4) : la convergence en loi de $n^{-1}\bigvee_{i=1}^{n}\xi_i$ est équivalente à la condition de variation régulière

$$n^{-1}\mathbf{P}(n^{-1}\xi \in \cdot) \xrightarrow{\hat{w}} \nu, \tag{1}$$

où ν est une mesure de Radon, et où le sens de la convergence de mesures $\xrightarrow{\hat{w}}$ est précisé dans l'introduction.

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Lorsque les trajectoires de ξ ne sont pas continues, mais appartiennent à l'espace $\mathbb{D}$ des fonctions cadlag de [0, 1] dans $\mathbb{R}$, seul un sens de l'équivalence précédente reste vrai. Le maximum normalisé ne peut converger sans que la condition (1) soit satisfaite, mais la réciproque est fausse. Ceci est lié au fait que l'application $\Phi : \mathbb{D}^2 \to \mathbb{D}$, définie par $\Phi(f,g) := f \vee g$, n'est pas continue ($\mathbb{D}$ étant muni de la topologie de Skorokhod). Dans un premier temps nous exhibons un exemple qui montre que l'équivalence n'a pas lieu en toute généralité. Ensuite nous montrons que l'application Φ est continue en tout point $(f,g) \in \mathbb{D}^2$ si f et g n'ont pas de points de discontinuité commun. Cela nous permet d'affirmer que si la condition de variation régulière (1) est satisfaite, avec ν telle que

$$\nu\big(\{f \in \mathbb{D},\, f(x^-) \neq f(x)\}\big) = 0, \quad \text{pour tout } x \in [0,1],$$

alors le maximum normalisé converge en loi dans $\mathbb{D}$.

1. Introduction

Let us begin with some notation. Denote by $\mathbb{C}$ the space of continuous functions $f : [0,1] \to \mathbb{R}$, and by $\mathbb{D}$ the space of cadlag functions $f : [0,1] \to \mathbb{R}$, equipped with the distance d, defined by

$$d(f_1, f_2) := \inf\big\{\epsilon > 0, \exists \lambda \in \Lambda, \|\lambda - \mathrm{Id}\|_\infty < \epsilon, \|f_1 - f_2 \circ \lambda\|_\infty < \epsilon\big\},$$

where Λ is the set of continuous strictly increasing functions $\lambda : [0,1] \to [0,1]$, satisfying $\lambda(0) = 0$ and $\lambda(1) = 1$. Denote $\mathbb{D}^+ = \{f \in \mathbb{D}, f \geqslant 0\}$, $S_{\mathbb{D}^+} = \{f \in \mathbb{D}^+, \|f\|_\infty = 1\}$, and define $\overline{\mathbb{D}}_0^+ := (0,\infty] \times S_{\mathbb{D}^+}$, where $(0,\infty]$ is equipped with the metric $\rho(x,y) = |1/x - 1/y|$. The bounded sets in $\overline{\mathbb{D}}_0^+$ are those which are bounded away from 0. We say that $\mu_n \xrightarrow{\hat{w}} \mu$ if $\mu_n(B) \to \mu(B)$, for all bounded sets B satisfying $\mu(\partial B) = 0$. If the space is locally compact, ŵ-convergence coincides with vague convergence.

For i.i.d. random elements $\xi, \xi_1, \xi_2, \ldots$, the link between regular variation of ξ and convergence of the normalized maximum $M_n = n^{-1} \bigvee_{i=1}^n \xi_i$ is known in finite dimension (see [5]). De Haan and Lin (see [3]) studied this problem in the more general case of stochastic processes. They obtained the following result (Theorem 2.4 of [3]):

Theorem 1.1. *Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. random elements in $\mathbb{D}^+ \setminus \{0\}$. Consider the three following assertions*:

(i) $n^{-1} \bigvee_{i=1}^n \xi_i \xrightarrow{\mathrm{d}} \eta$ *in* $\mathbb{D}^+ \setminus \{0\}$.
(ii) $\nu_n \xrightarrow{\hat{w}} \nu$ *in the space of Radon measures in* $\overline{\mathbb{D}}_0^+$, *where* $\nu_n(\cdot) := n\mathbf{P}(n^{-1}\xi \in \cdot)$.
(iii) $N_n \xrightarrow{\mathrm{d}} N$ *in the space of point processes in* $\overline{\mathbb{D}}_0^+$, *where* $N_n := \sum_{i=1}^n \epsilon_{\{n^{-1}\xi_i\}}$, *and N is a Poisson point process.*

Then the following implications hold: (i) $\Rightarrow$ (ii) *and* (ii) $\Leftrightarrow$ (iii). *Moreover, ν is the intensity measure of the point process N, and we have $\eta \overset{d}{=} \bigvee_{i=1}^\infty \zeta_i$, where $(\zeta_i)_{i\geqslant 1}$ are the points of N. Finally, if $\mathbf{P}(\eta \in \mathbb{C}) = 1$, then* (ii) $\Rightarrow$ (i).

The purpose of this Note is to show that the implication (ii) $\Rightarrow$ (i) is not true in general, and to give a condition under which it is true. A counterexample is given in Section 2. In Section 3 we study the continuity of the application $\Phi : (\mathbb{D}^2, d \times d) \to (\mathbb{D}, d)$, defined by $\Phi(f,g)(x) := (f \vee g)(x) = f(x) \vee g(x)$. Finally we conclude in Section 4 where we show that if the limit measure ν in (ii) satisfies

$$\nu\big(\{f \in \mathbb{D},\, f(x^-) \neq f(x)\}\big) = 0, \quad \text{for all } x \in [0,1], \tag{2}$$

then the implication (ii) $\Rightarrow$ (i) takes place.

2. A counterexample

Let X be the random function in $\mathbb{D}^+ \setminus \{0\}$ defined by

$$X := \big(b\mathbb{1}_{[\min(1/2+1/F,1),1]} + (1-b)\mathbb{1}_{[0,\max(0,1/2-1/F)[}\big)F \tag{3}$$

where b is a Bernoulli variable with parameter $1/2$, F has a unit Frechet distribution and is independent of b. The paths of X are step functions with a single step. The jump of X is close to $1/2$ when F takes high values. Let (F_i, b_i), $i \in \mathbb{N}$, be i.i.d. copies of (F, b) and define

$$X_i := \big(b_i\mathbb{1}_{[\min(1/2+1/F_i,1),1]} + (1-b_i)\mathbb{1}_{[0,\max(0,1/2-1/F_i)[}\big)F_i.$$

The following two propositions say that the process X satisfies assertion (ii) of Theorem 1.1, although assertion (i) is not satisfied:

Proposition 2.1. *X is regularly varying in $\overline{\mathbb{D}}_0^+$. More precisely, the following convergence takes place*:

$$\nu_n = n\mathbf{P}(n^{-1}X \in \cdot) \xrightarrow[n\to\infty]{\hat{\mathrm{w}}} \nu,$$

where ν is a Radon measure on $\overline{\mathbb{D}}_0^+$.

Proof. In order to use Theorem 10 from [4], we check the convergence of the finite-dimensional distributions, and tightness.

Let us begin with finite-dimensional distributions. Let $t_1, t_2, \ldots, t_k \in [0,1] - \{1/2\}$. Define

$$m^n_{t_1,\ldots,t_k}(\cdot) = n\mathbf{P}\big(n^{-1}\big(X(t_1), \ldots, X(t_k)\big) \in \cdot\big).$$

We will show that the sequence of measures $m^n_{t_1,\ldots,t_k}$ on $\overline{\mathbb{R}^k} \setminus \{0\}$ converges vaguely to a measure $m_{t_1,\ldots,t_k}$ such that $m_{t_1,\ldots,t_k}(\overline{\mathbb{R}^k} \setminus \mathbb{R}^k) = 0$. By construction of X, it is easy to show that for all $\mathbf{a} = (a_1, \ldots, a_k) \in (\mathbb{R}^{+*})^k$, $m^n_{t_1,\ldots,t_k}([\mathbf{a}, +\infty])$ converges to

$$\frac{1}{2}\max(a_1, \ldots, a_k)^{-\alpha}(\mathbb{1}_{\max(t_1,\ldots,t_k)<1/2} + \mathbb{1}_{\min(t_1,\ldots,t_k)>1/2}) =: m_{t_1,\ldots,t_k}\big([\mathbf{a}, +\infty]\big).$$

This is sufficient to characterize the measure $m_{t_1,\ldots,t_k}$, and to prove the convergence vague.

To prove tightness, we must show that for all $\epsilon > 0$ and $\eta > 0$, there exists $\delta > 0$, such that

$$\lim_n n\mathbf{P}\big(w''(X, \delta) \geqslant n\epsilon\big) \leqslant \eta,$$
$$\lim_n n\mathbf{P}\big(w\big(X, [0, \delta)\big) \geqslant n\epsilon\big) \leqslant \eta,$$
$$\lim_n n\mathbf{P}\big(w\big(X, [1-\delta, 1)\big) \geqslant n\epsilon\big) \leqslant \eta.$$

By observing that the paths of X are step functions with a single step, these three conditions are easily checked, so the proof is complete. □

Proposition 2.2. *Let $M_n := n^{-1}\bigvee_{i=1}^n X_i$. For all $n \geqslant 1$, let P_n denote the distribution of M_n. Then the sequence $(P_n)_{n\geqslant 1}$ is not tight in the space of probability measures on $\mathbb{D}$.*

Proof. We will show that there exist $\epsilon > 0$ and $\eta > 0$, such that for all $\delta > 0$,

$$\lim_n \inf\big\{P_n\big(f \in \mathbb{D}, w''_f(\delta) \geqslant \epsilon\big)\big\} \geqslant \eta, \tag{4}$$

where

$$w''_f(\delta) := \sup_{t_1 \leqslant t \leqslant t_2 \text{ et } t_2 - t_1 \leqslant \delta} \min\big(\big|f(t) - f(t_1)\big|, \big|f(t_2) - f(t)\big|\big).$$

This will prove our claim by applying Theorem 14.4 of [1].

Define $A_{n,0} = \{i, 1 \leqslant i \leqslant n,\ b_i = 0\}$, $A_{n,1} = \{i, 1 \leqslant i \leqslant n,\ b_i = 1\}$, $a_{n,0} = |A_{n,0}|$, and $a_{n,1} = |A_{n,1}|$. For $k = 0, 1$, define $i_{n,k} = \min\{i \in A_{n,k}, F_i = \max(F_j, j \in A_{n,k})\}$ if $A_{n,k} \neq \emptyset$, and $i_{n,k} = 0$ else. With these notations, we have

$$M_n = \begin{cases} n^{-1}F_{i_{n,1}}\mathbf{1}_{[\min(1/2+1/F_{i_{n,1}},1),1]} & \text{if } A_{n,0} = \emptyset; \\ n^{-1}F_{i_{n,0}}\mathbf{1}_{[0,\max(0,1/2-1/F_{i_{n,0}})]} & \text{if } A_{n,1} = \emptyset; \\ n^{-1}F_{i_{n,1}}\mathbf{1}_{[\min(1/2+1/F_{i_{n,1}},1),1]} + n^{-1}F_{i_{n,0}}\mathbf{1}_{[0,\max(0,1/2-1/F_{i_{n,0}})[} & \text{else.} \end{cases}$$

By construction, $M_n(1/2) = 0$ a.s. Let $\delta \in (0, 1/2)$ and $\eta \in (0, 1/2)$. For all $\epsilon > 0$, we have

$$\underbrace{\big\{M_n(1/2 + \delta/2) > \epsilon\big\}}_{E_{n,0}} \cap \underbrace{\big\{M_n(1/2 - \delta/2) > \epsilon\big\}}_{E_{n,1}} \subset \big\{w''_{M_n}(\delta) \geqslant \epsilon\big\}.$$

Obviously, $\mathbf{P}(E_{n,0}) = \mathbf{P}(E_{n,1})$. $E_{n,0}$ can be decomposed as $E_{n,0} = \{A_{n,0} \neq \emptyset\} \cap \{(F_{i_{n,0}})^{-1} < \delta/2\} \cap \{n^{-1} F_{i_{n,0}} > \epsilon\}$. Since $a_{n,0}/n \to 1/2$ a.s., it follows that

$$\mathbf{P}(E_{n,0}) \xrightarrow{n\to\infty} \mathbf{P}(F > 2\epsilon). \tag{5}$$

Let $\epsilon > 0$, such that $\mathbf{P}(F > 2\epsilon) > 1/2 + \eta$. Then

$$P_n\big(f \in \mathbb{D}, w''_f(\delta) \geqslant \epsilon\big) \geqslant \mathbf{P}(E_{n,0} \cap E_{n,1}) \geqslant \mathbf{P}(E_{n,0}) + \mathbf{P}(E_{n,1}) - 1. \tag{6}$$

Finally, (5) and (6) imply (4). □

3. Continuity of the maximum in $\mathbb{D}$

Recall that the application Φ is defined on $\mathbb{D}^2$ by $\Phi(f, g) := f \vee g$.

Proposition 3.1. *Φ is continuous at each point of the set*

$$\big\{(f_1, f_2) \in \mathbb{D}^2, \operatorname{disc}(f_1) \cap \operatorname{disc}(f_2) = \emptyset\big\},$$

where $\operatorname{disc}(f)$ *denotes the set* $\{x \in [0,1], f(x^-) \neq f(x)\}$.

In order to prove the proposition, we begin with three lemmas. Let $\mathcal{E} \subset \mathbb{D}$ be the set of simple functions. We know that $\mathcal{E}$ is dense in $\mathbb{D}$ endowed with the L_∞ norm.

Lemma 3.2. *Consider $\tilde{\Phi}$ the restriction of Φ to $\mathcal{E}^2$. Then $\tilde{\Phi}$ is continuous at each point of the set*

$$\big\{(g_1, g_2) \in \mathcal{E}^2, \operatorname{disc}(g_1) \cap \operatorname{disc}(g_2) = \emptyset\big\}.$$

Proof of Lemma 3.2. Let $g_1, g_2 \in \mathcal{E}$ having no common discontinuity point. Let A be the set of all discontinuity points of g_1 and g_2, and define $l := \min\{a - b, a \in A, b \in A, a \neq b\}$. Let ϵ, $0 < \epsilon < l/3$.

Let $\tilde{g_1}, \tilde{g_2} \in \mathcal{E}$ such that $d(g_1, \tilde{g_1}) \leqslant \epsilon$ and $d(g_2, \tilde{g_2}) \leqslant \epsilon$. By observing that small perturbations of the time do not change the order of the jumps, we can easily prove that there exists $\lambda \in \Lambda$ such that $\|\mathrm{Id} - \lambda\|_\infty \leqslant \epsilon$, $\|g_1 \circ \lambda - \tilde{g_1}\|_\infty \leqslant \epsilon$, $\|g_2 \circ \lambda - \tilde{g_2}\|_\infty \leqslant \epsilon$ and $\Phi(g_1 \circ \lambda, g_2 \circ \lambda) = \Phi(g_1, g_2) \circ \lambda$. Thus

$$\begin{aligned} d\big(\Phi(g_1, g_2), \Phi(\tilde{g_1}, \tilde{g_2})\big) &\leqslant d\big(\Phi(g_1, g_2), \Phi(g_1 \circ \lambda, g_2 \circ \lambda)\big) + d\big(\Phi(g_1 \circ \lambda, g_2 \circ \lambda), \Phi(\tilde{g_1}, \tilde{g_2})\big) \\ &\leqslant d\big(\Phi(g_1, g_2), \Phi(g_1, g_2) \circ \lambda\big) + \big\|\Phi(g_1 \circ \lambda, g_2 \circ \lambda) - \Phi(\tilde{g_1}, \tilde{g_2})\big\|_\infty. \end{aligned}$$

By using the fact that for all $f_1, f_2, f_3, f_4 \in \mathbb{D}$, $\|\Phi(f_1, f_2) - \Phi(f_3, f_4)\|_\infty \leqslant \|f_1 - f_3\|_\infty \vee \|f_2 - f_4\|_\infty$, we can verify that $d(\Phi(g_1, g_2), \Phi(\tilde{g_1}, \tilde{g_2})) \leqslant 2\epsilon$. □

Lemma 3.3. *Let $(f_1, f_2) \in \mathbb{D}^2$, such that* $\operatorname{disc}(f_1) \cap \operatorname{disc}(f_2) = \emptyset$, *and let* $\epsilon > 0$. *Then there exists* $(g_1, g_2) \in \mathcal{E}^2$, *satisfying* $\operatorname{disc}(g_1) \cap \operatorname{disc}(g_2) = \emptyset$, *and such that*

$$\|f_1 - g_1\| < \epsilon \quad \textit{and} \quad \|f_2 - g_2\| < \epsilon.$$

Proof of Lemma 3.3. Let $(g_1, \tilde{g_2}) \in \mathcal{E}^2$ be such that $\|f_1 - g_1\| < \epsilon$, and $\|f_2 - \tilde{g_2}\| < \epsilon/2$. Since g_1 and $\tilde{g_2}$ have only a finite number of discontinuity points, we can find a function $\lambda \in \Lambda$, $\|\mathrm{Id} - \lambda\|_\infty < \epsilon/2$, such that g_1 and $g_2 := \tilde{g_2} \circ \lambda$ have no common discontinuity points. □

Lemma 3.4. *Let $f \in \mathbb{D}$, $a > 0$, and $r > 0$. Let $g \in \mathcal{E}$ satisfying $\|f - g\|_\infty < a$. Let $\tilde{f} \in \mathbb{D}$ satisfying $D(f, \tilde{f}) < r$. Then there exists $\tilde{g} \in \mathcal{E}$ such that*

$$\|\tilde{f} - \tilde{g}\| < r + a \quad \textit{and} \quad d(g, \tilde{g}) < r.$$

Proof of Lemma 3.4. Let $\lambda \in \Lambda$ be such that $\|\mathrm{Id} - \lambda\|_\infty \leqslant r$ and $\|\tilde{f} - f \circ \lambda\|_\infty \leqslant r$. It suffices to take $\tilde{g} := g \circ \lambda$. □

Proof of Proposition 3.1. Let $(f_1, f_2) \in \mathbb{D}^2$ satisfying $\operatorname{disc}(f_1) \cap \operatorname{disc}(f_2) = \emptyset$, and let $\epsilon > 0$. Let (cf. Lemma 3.3) $(g_1, g_2) \in \mathcal{E}^2$ be such that $\operatorname{disc}(g_1) \cap \operatorname{disc}(g_2) = \emptyset$, $\|g_1 - f_1\|_\infty < \epsilon/4$, and $\|g_2 - f_2\|_\infty < \epsilon/4$. Let $r_\epsilon > 0$ (cf. Lemma 3.2) be such that for every $(\tilde{g_1}, \tilde{g_2}) \in \mathcal{E}^2$ satisfying $d(g_1, \tilde{g_1}) \vee d(g_2, \tilde{g_2}) < r_\epsilon$, we have

$$d\big(\Phi(g_1,g_2),\Phi(\tilde{g_1},\tilde{g_2})\big)<\epsilon/2.$$

Let $(\tilde{f}_1,\tilde{f}_2)\in\mathbb{D}^2$ be such that $d(f_1,\tilde{f}_1)\vee d(f_2,\tilde{f}_2)<r_\epsilon$. We will show that $d(\Phi(f_1,f_2),\Phi(\tilde{f}_1,\tilde{f}_2))<\epsilon$, then the proof will be complete. Let (cf. Lemma 3.4 with $r=r_\epsilon<\epsilon/8$, and $a=\epsilon/8$) $(\tilde{g_1},\tilde{g_2})\in\mathcal{E}^2$, such that

$$\|\tilde{g_1}-\tilde{f}_1\|<\epsilon/4 \quad\text{and}\quad d(g_1,\tilde{g_1})<r_\epsilon,$$
$$\|\tilde{g_2}-\tilde{f}_2\|<\epsilon/4 \quad\text{and}\quad d(g_2,\tilde{g_2})<r_\epsilon.$$

By writing

$$d\big(\Phi(f_1,f_2),\Phi(\tilde{f}_1,\tilde{f}_2)\big)\leqslant d\big(\Phi(f_1,f_2),\Phi(g_1,g_2)\big)+d\big(\Phi(g_1,g_2),\Phi(\tilde{g_1},\tilde{g_2})\big)+d\big(\Phi(\tilde{g_1},\tilde{g_2}),\Phi(\tilde{f}_1,\tilde{f}_2)\big),$$

we can conclude that $d(\Phi(f_1,f_2),\Phi(\tilde{f}_1,\tilde{f}_2))\leqslant\epsilon/8+\epsilon/2+\epsilon/4<\epsilon$, and the result follows. □

The following two corollaries are straightforward consequences of Proposition 3.1:

Corollary 3.5. *The application Φ_n from*

$$\bigcup_{n\geqslant1}\big\{(f_1,\dots,f_n)\in\mathbb{D}^n,\operatorname{disc}(f_i)\cap\operatorname{disc}(f_j)=\emptyset,1\leqslant i\neq j\leqslant n\big\}$$

to $\mathbb{D}$, defined by $\Phi_n((f_1,\dots,f_n))=\bigvee_{i=1}^n f_i$, is continuous.

Corollary 3.6. *Consider the application, which as a finite point measure on $\mathbb{D}$, $m=\sum_{i=1}^n\epsilon_{f_i}$, associates $\bigvee_{i=1}^n f_i$. A sufficient condition for this application to be continuous at $m=\sum_{i=1}^n f_i$, is that for all $1\leqslant i\neq j\leqslant n$, we have* $\operatorname{disc}(f_i)\cap\operatorname{disc}(f_j)=\emptyset$.

4. Extension of the implication (ii) ⇒ (i)

The following result gives a condition under which regular variation of the process implies convergence of the normalized maximum. This generalizes the well-known result in finite dimension case and for continuous processes. For application, see for instance [2].

Proposition 4.1. *The notations are the same as in Theorem* 1.1. *Suppose that assertion* (ii) *is true. If, for all $x\in[0,1]$,*

$$\nu\big(\{f\in\mathbb{D},f(x^-)\neq f(x)\}\big)=0, \tag{7}$$

then assertion (i) *is true.*

Condition (7) means that there exists no deterministic point such that the process X conditioned to take high values jumps near this point with strictly positive probability. This is the case of Levy processes (see the example).

Proof. The proof of Theorem 1.1 in [3] can easily be generalized thanks to Corollary 3.6. Indeed, suppose that assertion (ii) is true and that condition (7) holds. By Theorem 1.1, N_n converges to N in the space of point processes. Therefore $N_{n,\epsilon}$ converges to N_ϵ in the space of finite point processes, where

$$N_{n,\epsilon}:=\sum_{i=1}^n\epsilon_{\{n^{-1}\xi_i\}}\mathbb{1}_{\{n^{-1}\|\xi_i\|_\infty\geqslant\epsilon\}}\quad\text{and}\quad N_\epsilon:=\sum_{i\geqslant1}\epsilon_{\{n^{-1}\zeta_i\}}\mathbb{1}_{\{n^{-1}\|\zeta_i\|_\infty\geqslant\epsilon\}}.$$

Using condition (7), we see that with probability 1, the points of the support of N_ϵ satisfy the assumption of Corollary 3.6. By the continuity theorem, we obtain

$$\frac{1}{n}\bigvee_{i=1}^n\xi_i\vee\epsilon\xrightarrow{\text{d}}\bigvee_{i\geqslant1}\zeta_i\vee\epsilon=\eta\vee\epsilon.$$

Letting ϵ go to 0, we get the result. □

Example. Let $X = (X_t)_{t \in [0,1]}$ be a Levy process. Suppose that X_1 satisfies $n\mathbf{P}(n^{-1}X_1 \in \cdot) \xrightarrow{\mathrm{v}} \mu_1$, where μ_1 is an 1-homogeneous Radon measure on $\overline{\mathbb{R}} \setminus \{0\}$. Then (cf. Example 17 in [4]) X satisfies assertion (ii), and the limit measure ν can be written as $\nu = \mu_1 \times \sigma$ (recall that $\overline{\mathbb{D}}_0^+ = (0, \infty] \times S_{\mathbb{D}^+}$), where σ is given by

$$\sigma(\cdot) = \mathbf{P}(Z\mathbb{1}_{[V,1]} \in \cdot),$$

where the distribution of Z is the spectral measure of X_1, and V is uniform on $[0, 1]$, such that Z and V are independent. Then condition (7) is checked. Let $(X^{(i)})_{i \geqslant 1}$ be a sequence of i.i.d. copies of X. By Proposition 4.1, $n^{-1} \bigvee_{i=1}^n X^{(i)}$ converge to a limit process η. We can see that η is a nondecreasing pure jump process.

References

[1] P. Billingsley, Convergence of Probability Measures, John Wiley & Sons Inc., New York, 1968.

[2] R.A. Davis, T. Mikosch, Extreme value theory for space–time processes with heavy tailed distributions, Stochastic Process. Appl. 2006, DOI: 10.1016/j.spa.2007.06.001.

[3] L. de Haan, T. Lin, On convergence toward an extreme value distribution in $C[0, 1]$, Ann. Probab. 29 (1) (2001) 467–483.

[4] H. Hult, F. Lindskog, On regular variation for infinitely divisible random vectors and additive processes, Adv. Appl. Probab. 38 (1) (2006) 134–148.

[5] S.I. Resnick, Extreme Values, Regular Variation, and Point Processes, Applied Probability, A series of the Applied Probability Trust, vol. 4, Springer-Verlag, New York, 1987.