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Partial Differential Equations

# Existence of topologically cylindrical shocks

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**Abstract**

In this Note the multidimensional stability of cylindrical shock profiles and the existence of a nearby perturbed structure is presented for the full Euler equations. This provides an example of a *nonplanar* structure for which the uniform Kreiss–Lopatinski–Majda stability condition can be explicitly verified. ***To cite this article: N. Costanzino, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


**Résumé**

**Existence des chocs topologiquement cylindrique.** Dans cette Note la stabilité multidimensionnelle des chocs cylindrique et de l'existence d'une structure perturbée voisine est présentée. Ceci fournit un exemple explicite d'une structure *non planaire*pour laquelle la condition de stabilité uniforme de Kreiss–Lopatinsky–Majda est satisfaite. ***Pour citer cet article : N. Costanzino, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


## 1. Introduction

In the seminal work [5,6], the nonlinear multidimensional stability of planar shocks of hyperbolic systems is proved under a natural assumption on a related linear constant coefficient problem. This assumption, often called the uniform Kreiss–Lopatinski–Majda condition, can be formulated in terms of nonvanishing in the right-hand complex plane of a particular determinant called the Lopatinski determinant. This is essentially a spectral condition whose fundamental property is that it allows one to obtain a maximal $L^2$ estimate for the linear frozen coefficient problem, and in turn a maximal $L^2$ estimate for the linear variable coefficient problem via Kreiss symmetrizers and the paradifferential calculus (cf. [7,8]). This condition has been explicitly verified in the case of *planar* gas dynamical shocks (cf. [5] for the barotropic Euler equations and [4] for the full Euler equations). In particular for an ideal gas, the Lopatinski determinant for planar shocks never vanishes in the right-half complex plane. However, if the initial shock is not planar but rather lies on a smooth *compact* hypersurface $\mathcal{M}_0$, then the uniform Kreiss–Lopatinski–Majda condition must be verified at each point $\alpha_0 \in \mathcal{M}_0$. In other words, for each point $\alpha_0 \in \mathcal{M}_0$, the planar shock given by $T_{\alpha_0}\mathcal{M}_0$ must satisfy the uniform Kreiss–Lopatinski–Majda condition. In this sense, stability of *curved* shocks reduces to the stability of a family of planar shocks. In general it is impractical to verify the condition for every point on $\mathcal{M}_0$ and

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hence so far the only explicit examples of Lopatinski determinants have been for planar shocks. Here we propose an alternative but related approach, which reduces to constructing a Lopatinski determinant for planar shocks of systems whose fluxes depend explicitly on the spatial variables. The simplest such case, namely cylindrical shocks, is presented here. Details of the proof in a more general setting can be found in [1]. One difference between cylindrical and planar shocks is that unlike planar shocks where the solution is *constant* on either side of a perturbed planar surface, a simple computation shows that it is not possible to construct radially symmetric solutions which are constant on either side of $r=\bar{r}(t)$. However, it is possible to construct radially symmetric *profiles* depending on $r$ with a discontinuity across $r=r(t)$. Here we focus on a setup which admits *stationary* radially symmetric profiles $\bar{U}(r)$ with a discontinuity on $r=\bar{r}=$ constant. This is achieved by considering the equations between two concentric cylinders of radius $a>0$ and $b>a$, with Dirichlet boundary conditions at $r=a$ and $r=b$. Note that in this setup the shock surface is the infinite cylinder, so that the precise theorem as stated in [6] does not cover this particular noncompact setting. We nevertheless define a Lopatinski determinant for cylindrical shocks $\Delta_{\mathbb{S}^1}$, and construct solutions that are small multidimensional perturbations of the reference radial solutions $\bar{U}(r)$ under the condition that $\Delta_{\mathbb{S}^1}$ has no unstable zeroes. Just as in [2], even though the reference solution $\bar{U}(r)$ is a profile, it is merely the linearization about the constant states at the right and left of the jump discontinuity that determines the short-time existence of a nearby perturbed structure.

## 2. Formulation of the problem and main result

The full Euler equations in cylindrical coordinates can be written in conservative form as $\mathbb{L}(r,U)=G(r,U)$ where

$$\mathbb{L}(r,U)=\partial_t F^t(U)+\partial_r F^r(U)+\partial_\theta F^\theta(r,U)+\partial_z F^z(U),\quad U\in\mathbb{R}^5 \tag{1}$$

and the fluxes are given by

$$F^t(U)=\begin{pmatrix}\rho\\ \rho u_r\\ \rho u_\theta\\ \rho u_z\\ \rho E\end{pmatrix},\qquad F^r(U)=\begin{pmatrix}\rho u_r\\ \rho u_r^2+P\\ \rho u_r u_\theta\\ \rho u_r u_z\\ (\rho E+P)u_r\end{pmatrix},$$

$$F^\theta(r,U)=\frac{1}{r}\begin{pmatrix}\rho u_\theta\\ \rho u_r u_\theta\\ \rho u_\theta^2+P\\ \rho u_\theta u_z\\ (\rho E+P)u_\theta\end{pmatrix},\qquad F^z(U)=\begin{pmatrix}\rho u_z\\ \rho u_r u_z\\ \rho u_\theta u_z\\ \rho u_z^2+P\\ (\rho E+P)u_z\end{pmatrix}$$

and $G(r,U)=-\frac{1}{r}(\rho u_r,\rho(u_r^2-u_\theta^2),2\rho u_r u_\theta,\rho u_r u_z,(\rho E+P)u_r)^{\mathrm{tr}}$. Here $P$ is the pressure and $E=e+\frac{1}{2}|U|^2$ is the energy. Note that in $F^\theta$ depends *explicitly* on the spatial variable $r$, but not on $\theta$. The solutions we construct will be small perturbations of a radially symmetric reference solution $\bar{U}(r)$. We consider (1) in

$$C_a^b:=\{(r,\theta,z)\colon\ a<r<b,\ -\pi\leqslant\theta\leqslant\pi,\ -\infty<z<+\infty\},$$

the space bounded between two concentric cylinders of radius $a$ and $b$. This particular setup allows us to construct *stationary* shock profiles $\bar{U}(r)$ to (1) which solve

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}F^r(\bar{U})=G(r,\bar{U}),\quad \text{for } r\in[a,\bar{r})\cup(\bar{r},b],\\ &\big[F^r(U)\big]=0,\quad \text{on } r=\bar{r},\\ &\bar{U}|_{r=a}=U_a\quad\text{and}\quad \bar{U}|_{r=b}=U_b.\end{aligned}\right\} \tag{2}$$

Stationary shock profiles solving (2) have been constructed in [3] for both the barotropic and full Euler equations. Suppose that $U(t,r,\theta,z)=\bar{U}(r)+\mathcal{U}(t,r,\theta,z)$ is smooth on either side of a codimension-one surface described by $\mathcal{S}:=\{(t,r,\theta,z)\in\mathbb{R}\times\mathbb{R}_+\times[-\pi,\pi]\times\mathbb{R}\colon r=R(t,\theta,z)$ and that $a<R(t,\theta,z)<b$ for all $(t,\theta,z)\in[0,T]\times[-\pi,\pi]\times\mathbb{R}$. Furthermore, suppose $U$ is smooth in $\{a<r<R(t,\theta,z)\}\cup\{R(t,\theta,z)<r<b\}$. Then $U$ is a distributional solution

to (1) if and only if it is a classical solution of (1) in $\{a < r < R(t,\theta,z)\} \cup \{R(t,\theta,z) < r < b\}$ and the Rankine–Hugoniot conditions $\mathbb{B}(r,U,dR) = 0$ are satisfied on $r = R(t,\theta,z)$, where

$$\mathbb{B}(r,U,\mathrm{d}R) = \partial_t R\big[F^t(U)\big] + \frac{1}{r}\partial_\theta R\big[F^\theta(U)\big] + \partial_z R\big[F^z(U)\big] - \big[F^r(U)\big].$$

Hence topologically cylindrical shocks are solutions to the initial free boundary problem for $(U,R)$ with Dirichlet boundary conditions at $r = a$ and $r = b$, namely,

$$\left.\begin{aligned} &\mathbb{L}(r,U) = G(r,U) \quad \text{in } [0,t] \times C_a^b \setminus \big\{r = R(t,\theta,z)\big\},\\ &\mathbb{B}(r,U,\mathrm{d}R) = 0 \quad \text{on } [0,T] \times \big\{r = R(t,\theta,z)\big\},\\ &U|_{r=a} = U_a, \quad U|_{r=b} = U_b, \quad U|_{t=0} = U_0. \end{aligned}\right\} \tag{3}$$

In light of (2) we solve (3) with $(U(t,r,\theta,z), R(t,\theta,z)) = (\bar{U}(r) + \mathcal{U}(t,r,\theta,z), \bar{r} + \mathcal{R}(t,\theta,z))$.

**Theorem 2.1** *(Existence of Topologically Cylindrical Shocks). For any $T > 0$ define $\mathcal{C}_T^\pm := \{[0,T] \times C_a^b\} \cap \{\pm r > R(t,\theta,z)\}$ and $\mathcal{C}_T^0 := \{[0,T] \times C_a^b\} \cap \{r = R(t,\theta,z)\}$. Then for any initial perturbation $\mathcal{U}_0 = (U_0 - \bar{U}) \in H^{k+1}$, $k > d/2 + 2$, satisfying the compatibility conditions of* [6] *and having support away from the boundaries $r = a$ and $r = b$, there exists a $T > 0$ small enough such that* (3) *has a distributional solution $(U(t,r,\theta,z), R(t,\theta,z))$, not necessarily unique, such that*

$$U^\pm - \bar{U} \in H^k\big(\mathcal{C}_T^\pm\big), \quad \big(U^\pm - \bar{U}\big)\big|_{r=R(t,\theta,z)} \in H^{k+1}\big(\mathcal{C}_T^0\big), \quad R \in H^{k+1}\big(\mathcal{C}_T^0\big). \tag{4}$$

*The existence time $T$ depends on the norm of the initial perturbation.*

**Sketch of proof.** The solution to (3) can be constructed by Picard iteration once we get a maximal $L^2$ estimate on the linearization of (3) about $(\bar{U}, \bar{r})$. The estimate can be decomposed into one where the coefficients of the linearization are away from the shock and one near the shock. The estimate away from the shock is obtained from the Friedrichs theory for smooth coefficients since (1) has a Friedrichs symmetrizer. For the estimate near the shock, we define the constant states $U_\pm$ as $U_\pm = \lim_{r \to \bar{r}\pm} \bar{U}(r)$ so that since (1) is written in conservative form it is clear that

$$U_{\text{shock}} := \begin{cases} U_-, & r \in (a, \bar{r}),\\ U_+, & r \in (\bar{r}, b) \end{cases} \tag{5}$$

satisfies (3) if the source term is set to zero and we ignore the boundary conditions. Hence after mapping (3) to a fixed-boundary transmission problem, dropping the geometric source term $G(r,U)$, and linearizing about $U_{\text{shock}}$, we are lead to the constant coefficient problem $\mathcal{L}(\bar{r}, U_\pm)\mathcal{U} = 0$, $\mathcal{B}(U_\pm, \bar{r})(\mathcal{U}, \mathcal{R}) = 0$ where

$$\begin{aligned} &\mathcal{L}(\bar{r}, U_\pm)\mathcal{U} := \partial_t \mathcal{U} + A^r(U_\pm)\partial_r \mathcal{U} + \frac{1}{\bar{r}} A^\theta(U_\pm)\partial_\theta \mathcal{U} + A^z(U_\pm)\partial_z \mathcal{U},\\ &\mathcal{B}(U_\pm, \bar{r})(\mathcal{U}, \mathcal{R}) = \partial_t \mathcal{R}\big[F^t(U_{\text{shock}})\big] + \frac{1}{\bar{r}}\partial_\theta \mathcal{R}\big[F^\theta(U_{\text{shock}})\big] + \partial_z \mathcal{R}\big[F^z(U_{\text{shock}})\big] - \big[A^r(U_{\text{shock}})\mathcal{U}\big]. \end{aligned} \tag{6}$$

The main idea here is that an estimate on the system obtained by freezing coefficients at the shock location will yield an estimate for the variable coefficient problem for coefficients which are small perturbations of the frozen coefficients. Taking the Laplace–Fourier transform in the tangential variables $(t,\theta,z)$ yields the system of ordinary differential equations in the radial variable $r$,

$$\partial_r \hat{U} = -\mathcal{H}_\pm(\lambda, \eta_\theta, \eta_z; \kappa)\hat{U} = -\big(A^r(U_\pm)\big)^{-1}\big\{\lambda A^t(U_\pm) + \mathrm{i}\kappa\eta_\theta A^\theta(U_\pm) + \mathrm{i}\eta_z A^z(U_\pm)\big\}\hat{U} \tag{7}$$

where the tangential frequencies $(\lambda, \eta_\theta, \eta_z)$ as well as $\kappa$ are now parameters. Here $\kappa := \frac{1}{\bar{r}}$ is the curvature of the underlying *unperturbed* stationary cylindrical shock satisfying (2). The frequencies are such that $\lambda \in \mathbb{C}$, $\eta_\theta \in \mathbb{Z}$, $\eta_z \in \mathbb{R}$. Define a rescaled angular frequency $\bar{\eta}_\theta := \kappa\eta_\theta$ and set $\Sigma^1 := \{(\lambda, \bar{\eta}_\theta, \eta_z) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{R}\colon \Re\lambda > 0\}$. The uniform Kreiss–Lopatinski–Majda condition is now phrased in terms of nonexistence of solutions to (7) with $\Re\lambda > 0$ through nonvanishing of the Lopatinski determinant on $\Sigma^1$.

From [3] $U_{\text{shock}}$ has Lax shock structure so we can proceed to define a Lopatinski determinant as

$$\Delta_{\mathbb{S}^1}(\lambda, \bar{\eta}_\theta, \eta_z) = \det\big(b(\lambda, \bar{\eta}_\theta, \eta_z), r_-^1(\lambda, \bar{\eta}_\theta, \eta_z), r_-^2(\lambda, \bar{\eta}_\theta, \eta_z), r_-^3(\lambda, \bar{\eta}_\theta, \eta_z), r_-^4(\lambda, \bar{\eta}_\theta, \eta_z)\big) \tag{8}$$

where $b = \lambda[F^t(U_{\text{shock}})] + i\bar{\eta}_\theta[F^\theta(U_{\text{shock}})] + i\eta_z[F^z(U_{\text{shock}})]$ and $\{r_-^j\}_{j=1}^4$ span appropriate stable/unstable subspaces associated to (7) (cf. [1] for details). Then the Kreiss–Lopatinski–Majda condition is equivalent to requiring $\Delta_{\mathbb{S}^1}(\lambda, \bar{\eta}_\theta, \eta_z) \neq 0$ for every $(\lambda, \bar{\eta}_\theta, \eta_z) \in \Sigma^1$. □

**Lemma 2.2** *(Lopatinski Determinant for Cylindrical Shocks). Let $\Delta_{\mathbb{S}^1}(\lambda, \bar{\eta}_\theta, \eta_z)$ be the Lopatinski determinant associated to the shock discontinuity of the radially symmetric profile $\bar{U}(r)$. Then the zero set of $\Delta_{\mathbb{S}^1}$ on $\Sigma_1$ coincides with the zero set of $\Delta_E$, where $\Delta_E$ is the Lopatinski determinant for a planar gas dynamical extreme Euler shock computed in* [4]. *In particular for an ideal gas $\Delta_{\mathbb{S}^1}$ does not vanish in $\Sigma^1$.*

From Lemma 2.2 we can proceed as in [2,6–8] with appropriate modifications due to the discrete nature of the tangential frequency $\bar{\eta}_\theta$. Furthermore initial data is taken to have support away from the boundaries $r = a$ and $r = b$ so that by finite speed of propagation, the boundary conditions will be satisfied on $[0, T]$ for $T$ small enough.

Finally we remark that the stability conditions are phrased in terms of hydrodynamic quantities such as Mach number and compression ratio, and since these depend on the shock location $\bar{r}$ of the radial profile $\bar{U}(r)$, the nonlinear stability for any particular gas law (other than ideal gas) ostensibly may differ in the cylindrical case from that of the planar case.

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