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On the ARCH model with random coefficients

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Abstract

In the usual ARCH model, the coefficients have a degenerate distribution, and it is thus constant over realizations. In this Note we introduce the ARCH model, involving coefficients that are independent random variables and may vary over realizations. Conditions for the existence of a stationary solution and conditions ensuring the existence of higher order moments are obtained. The covariance structures of such models are studied. ***To cite this article: A. Bibi, M. Bousseboua, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur le modèle ARCH avec des coefficients aléatoires. Contrairement aux modèles ARCH usuels, où les coefficients sont supposés constants, nous considérons dans cette Note, une classe de modèles ARCH à coefficients aléatoires. Les conditions assurant l'existence et l'unicité de solutions stationnaires ainsi que l'existence des moments d'ordre supérieur et la structure de covariance de cette classe de modèles sont étudiées. ***Pour citer cet article : A. Bibi, M. Bousseboua, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

The class of Autoregressive conditional heteroscedasticity of order p, denoted as $ARCH(p)$, and introduced firstly by Engle [2], is a class of nonlinear models that plays an important role in financial econometrics and has sometimes proved useful in modelling the residuals for time series models. In this Note, we consider the model

$$X_t = e_t\sigma_t \quad \text{and} \quad \sigma_t^2 = A_0 + \sum_{i=1}^{p} A_i X_{t-i}^2, \quad t \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\} \tag{1}$$

where the coefficients $A_0, \ldots, A_p$ are discrete random variables taking a finite number of values such that, almost surely ($a.s.$) $A_0 > 0$ and $A_i \geqslant 0$, $1 \leqslant i \leqslant p$, and where $(e_t)_{t\in\mathbb{Z}}$ is an $i.i.d.$ sequence such that $E\{e_t\} = E\{e_t^3\} = 0$, $\kappa_1 = 1$ and $\kappa_2 < +\infty$ where $\kappa_k = E\{e_t^{2k}\}$. This class of models is noted as $RC\text{-}ARCH(p)$ and we assume that $A_0, \ldots, A_p$ are mutually independent and independent of e_t for all t, and X_t is independent of e_s for all $s > t$.

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The *RC-ARCH*(p) models may exhibit spurious long memory properties and switching regimes, so they can be seen as *STAR*(p) models (see Dick et al. [3]) with eventually a large number of regimes, which is of little use in practice. Some other interesting motivations and recent developments can be found in Kazakevičius et al. [6], Klivečka [7] and Thavaneswaran [8]. In this Note, we study the existence and the uniqueness of stationary solutions of Eq. (1) and investigate the conditions ensuring the existence of higher order moments. The covariance structure of the squared process is given.

2. Existence of stationary solutions and higher moments for *RC-ARCH*(p)

In this Note, we are only interested in causal solutions i.e., solutions such that Y_t is $\Im_t$-measurable, where $\Im_t := \sigma(\xi_s, s \leqslant t)$. By setting $Y_t = X_t^2$, $\xi_t = e_t^2$ we obtain

$$Y_t = A_0\xi_t + \sum_{i=1}^{p} A_i\xi_t Y_{t-i}. \tag{2}$$

In the following, we will extensively use (2).

2.1. Stationary solution for a RC-ARCH(p)

Let us define the matrix $\mathbf{A}_t = (A_j\xi_t\delta_{i,1} + \delta_{i,j+1})_{1\leqslant i,j\leqslant p}$ where δ_{ij} is the Kronecker symbol and the vectors $\underline{Y}_t := (Y_t, \ldots, Y_{t-p+1})'_{p\times 1}$ and $\underline{b}_t := (\xi_t A_0, 0, \ldots, 0)'_{p\times 1}$. Then

$$\underline{Y}_t = \mathbf{A}_t \underline{Y}_{t-1} + \underline{b}_t \tag{3}$$

and $Y_t = C'\underline{Y}_t$ where $C := (1, 0, \ldots, 0)'$. Let $\|\cdot\|$ be the matrix norm inducted by any vectorial norm on $\mathbb{R}^p$ and $\log^+ x = \max\{\log x, 0\}$. It can be shown (cf. [4]) that if $E\{\log^+\|\mathbf{A}_0\|\} < +\infty$ then

$$\gamma_L := \lim\sup_{n\to\infty}\left\{E\left\{\frac{1}{n}\log\left\|\prod_{i=0}^{n}\mathbf{A}_{t-i}\right\|\right\}\right\}$$

always exists (may be infinite) and furthermore $\gamma_L \leqslant \gamma_{op}$ where $\gamma_{op} := E\{\log\|\prod_{i=1}^{p}\mathbf{A}_j\|\}$.

Theorem 1. *If $\gamma_{op} < 0$, then the series*

$$\underline{Y}_t = \sum_{k=1}^{\infty}\left\{\prod_{i=0}^{k-1}\mathbf{A}_{t-i}\right\}\underline{b}_{t-k} + \underline{b}_t \tag{4}$$

converges a.s. and the process $(Y_t)_{t\in\mathbb{Z}}$ defined as the first component of $(\underline{Y}_t)_{t\in\mathbb{Z}}$ is the unique ca. (2).

Example 1. Consider the *RC-ARCH*(2) and let $\|A\| := \sum_{i,j}|a_{ij}|$, then we have

$$\|\mathbf{A}_t\mathbf{A}_{t-1}\| = A_1^2\xi_t\xi_{t-1} + A_2\xi_t + A_1A_2\xi_t\xi_{t-1} + A_1\xi_{t-1} + A_2\xi_{t-1}.$$

Hence, $E\{\log\|\mathbf{A}_t\mathbf{A}_{t-1}\|\} < 0$ whenever $\|E\{\mathbf{A}^2\}\| < 1$ where $\mathbf{A} = E\{\mathbf{A}_t|\sigma\{A_1, A_2\}\}$ with $\sigma\{A_1, A_2\}$ is the σ-algebra generated by A_1, A_2. In general, for a *RC-ARCH*(p) if $\|E\{\mathbf{A}^p\}\| < 1$, then $\gamma_L < 0$ and hence the results of Theorem 1 follows.

It is worth noting that the criterion $\gamma_L < 0$ (even $\gamma_{op} < 0$) is of little use for checking of stationarity in applied works. Indeed, the stationary solution needs to have some moments to make an estimation theory possible and the criterion does not guarantee the existence of such moments. This leads to search for conditions on the structure of $(A_i)_{0\leqslant i\leqslant p}$ ensuring the existence of moments for the strict stationary solution for which $\gamma_L < 0$.

Proposition 1. *Let $\mathbf{A} = E\{\mathbf{A}_t|\Im_{\mathbf{A}}\}$ with $\Im_{\mathbf{A}} := \sigma(A_0, \ldots, A_p)$ and $\rho(\mathbf{A})$ the maximum modulus of the eigenvalues of the matrix $\mathbf{A}$. Eq.* (3) *has a strictly stationary solution in $\mathbb{L}_1$ if and only if, almost surely*

$$\rho(\mathbf{A}) < 1. \tag{5}$$

Moreover, this solution is causal, unique and given by the series (4) *which converges almost surely and in mean.*

Proof. The proof follows from standard arguments (see Liu [5]). First, define a $\mathbb{R}^p$-valued stochastic process which will be used in the proof to generated a strictly stationary solution of (3). For any $(n,t) \in \mathbb{Z} \times \mathbb{Z}$, let

$$\underline{S}_n(t) = \mathbf{A}_t \underline{S}_{n-1}(t-1) + \underline{b}_t$$

if $n \geqslant 0$ and $\underline{0}$ otherwise. By a simple iteration, we see that $\underline{S}_n(t) = \sum_{k=1}^{n}\{\prod_{i=0}^{k-1}\mathbf{A}_{t-i}\}\underline{b}_{t-k} + \underline{b}_t$ and hence the process $(\underline{S}_n(t))_{t\in\mathbb{Z}}$ is strictly stationary for fixed $n \geqslant 0$. Therefore, since the elements of the matrices $\{\prod_{i=0}^{n-1}\mathbf{A}_{t-i}\}\underline{b}_{t-n}$ are nonnegative, we obtain for any $n \geqslant 1$,

$$E\left\{\left\|\prod_{i=0}^{n-1}\mathbf{A}_{t-i}\underline{b}_{t-n}\right\|\right\} = (1,\dots,1)E\{\mathbf{A}^n\}\underline{b}$$

where $\underline{b} = E\{\underline{b}_t\}$. On the other hand, the condition (5) implies that $E\{\mathbf{A}^n\}$ decays to 0 as $n \to \infty$. Thus $\underline{S}_n(t)$ converges *a.s.* and in $\mathbb{L}_1$ (by the Cauchy's criterion) as $n \to \infty$. Now, let $\underline{Y}_t = \lim_{n\to\infty}\underline{S}_n(t)$, it is easy to see that the process $(\underline{Y}_t)_{t\in\mathbb{Z}}$ is strictly stationary and satisfies (3). Conversely, from (3) we have for any $n \geqslant 1$

$$\underline{Y}_0 = \sum_{k=1}^{n-1}\left\{\prod_{i=0}^{k-1}\mathbf{A}_{-i}\right\}\underline{b}_{-k} + \left\{\prod_{i=0}^{n-1}\mathbf{A}_{-i}\right\}\underline{Y}_{-n}. \tag{6}$$

By taking the expectation of each side of (6), it follows that $E\{\underline{Y}_0\} > E\{\sum_{k=1}^{n-1}\mathbf{A}^k\underline{b}\}$; this shows that $E\{\sum_{k=1}^{\infty}\mathbf{A}^k\underline{b}\} < +\infty$. Therefore $\lim_{n\to\infty}\mathbf{A}^n\underline{b} = 0$ *a.s.* Let $(\underline{\alpha}_i)_{1\leqslant i\leqslant p}$ be the canonical basic of $\mathbb{R}^p$. Using the same argument that Bougerol and Picard [1], we obtain $\lim_{n\to\infty}\mathbf{A}^n\underline{\alpha}_i = 0$ *a.s.* for any $i = 1,\dots,p$ and this implies that $\lim_{n\to\infty}\mathbf{A}^n = 0$ *a.s.* which again implies that $\rho(\mathbf{A}) < 1$. The uniqueness of the solution is immediate. □

Corollary 1. *Under the Conditions of Proposition* 1*, the process* $(Y_t)_{t\in\mathbb{Z}}$ *defined as the first component of* $(\underline{Y}_t)_{t\in\mathbb{Z}}$ *is the unique, strictly stationary and causal solution in* $\mathbb{L}_1$ *of* (2)*.*

Remark 1. Since $\det(\lambda I_{(p)} - \mathbf{A}) = \lambda^p(1 - \sum_{i=1}^{p}A_i\lambda^{-i})$, then a strictly stationary solution $(Y_t)_{t\in\mathbb{Z}}$ of (2) exists in $\mathbb{L}_1$ if and only if $P(A_1 + A_2 + \cdots + A_p < 1) = 1$. However, the unique stationary solution of (2) is a weak white noise with variance $Var\{X_t\} = C'E\{(I_{(p)} - \mathbf{A})^{-1}\}\underline{b}$ where $I_{(p)}$ is identity matrix of order p.

2.2. Existence of higher order moments

In this subsection, we shall derive necessary and sufficient conditions for the finiteness of $E\{X_t^{2k}\}$ for any $k \geqslant 1$. The existence of $E\{X_t^{2k}\}$ reduce to the convergence of $(\underline{S}_n(t))_{n\geqslant 0}$ in $\mathbb{L}_k$, since as is shown in above subsection $(\underline{S}_n(t))_{n\geqslant 0}$ converge to $\underline{Y}_t$ in $\mathbb{L}_1$. The element of interest for determining $\mathbb{L}_k$-convergence is $V_n := E\{\|\underline{\Delta}_n(t)\|^k\}$ where $\underline{\Delta}_n(t) := S_n(t) - S_{n-1}(t)$. By using the properties of tensor product denoted by $\otimes$: $AB \otimes CD = (A\otimes C)(B\otimes D)$, $(AX)^{\otimes r} = A^{\otimes r}X^{\otimes r}$ and $\|A\|\|B\| = \|A\otimes B\| = \|B\otimes A\|$ we obtain

$$E\{\|\underline{\Delta}_n(t)\|^k\} = \left\|E\left\{\prod_{i=0}^{n-1}\mathbf{A}_{t-i}\underline{b}_{t-n}\right\}^{\otimes k}\right\| = \left\|E\left\{\prod_{i=0}^{n-1}\mathbf{A}_{t-i}^{\otimes k}\underline{b}_{t-n}^{\otimes k}\right\}\right\| \tag{7}$$

where $M^{\otimes r} := M\otimes M\otimes\cdots\otimes M$ r-time for every matrix M. Let $\mathbf{A}^{(k)} = E\{\mathbf{A}_t^{\otimes k}|\Im_{\mathbf{A}}\}$, $\underline{b}^{(k)} = E\{\underline{b}_t^{\otimes k}\}$, then from (7) we obtain $E\{\|\underline{\Delta}_n(t)\|^k\} \leqslant \|E\{(\mathbf{A}^{(k)})^n\}\underline{b}^{(k)}\|$. Hence

$$\|\underline{Y}_t\|_k = \{E\|\underline{Y}_t\|^k\}^{1/k} \leqslant \sum_{n\geqslant 0}\|\underline{\Delta}_n(t)\|_k \leqslant \left\{\sum_{n\geqslant 0}\|E(\mathbf{A}^{(k)})^n\|^{1/k}\right\}\|\underline{b}^{(k)}\|^{1/k}.$$

If $\rho(\mathbf{A}^{(k)}) < 1$, then $\|E(\mathbf{A}^{(k)})^n\|$ converge to zero (with exponential rate) as $n \to \infty$. Thus we have proved the following theorem

Theorem 2. *Assume that* $\kappa_k < \infty$ *and that almost surely*

$$\rho(\mathbf{A}^{(k)}) < 1. \tag{8}$$

Then, for all $t \in \mathbb{Z}$, the series $(\underline{Y}_t)_{t\in\mathbb{Z}}$ defined by (4) *converges in $\mathbb{L}_k$ and the process $(Y_t)_{t\in\mathbb{Z}}$ defined as the first component of $(\underline{Y}_t)_{t\in\mathbb{Z}}$ is strictly stationary and admits moments up to the order k. Conversely, the condition* (8) *is also necessary for the existence of strictly stationary process RC-ARCH(p) satisfies* (2) *such that $E\{X_t^{2k}\} < +\infty$.*

2.3. Covariance structure of RC-ARCH(p)

Assume that the process $(\underline{Y}_t)_{t\in\mathbb{Z}}$ satisfies the conditions of Theorem 2 for $k = 2$ and $\mathbf{\Gamma}(\tau) := E\{(\underline{Y}_t - \underline{\mu}_1)(\underline{Y}_{t-\tau} - \underline{\mu}_1)'\}$ its covariance function where $\underline{\mu}_1 := E\{\underline{Y}_t\}$. Let $\underline{\mu}_1^c := E\{\underline{Y}_t|\Im_{\mathbf{A}}\}$, $\mu_2^c(\tau) := E\{\underline{Y}_t\underline{Y}_{t-\tau}'|\Im_{\mathbf{A}}\}$. Firstly, since $\underline{Y}_t^{\otimes 2} = \underline{b}_t^{\otimes 2} + (\underline{b}_t \otimes \mathbf{A}_t + \mathbf{A}_t \otimes \underline{b}_t)\underline{Y}_{t-1} + \mathbf{A}_t^{\otimes 2}\underline{Y}_{t-1}^{\otimes 2}$ then $\underline{\mu}_1^c$ and $Vect\{\mu_2^c(0)\}$ is a solution to the equations $\underline{\mu}_1^c = (I_{(p)} - \mathbf{A})^{-1}\underline{b}$ and $Vect\{\mu_2^c(0)\} = (I_{(p^2)} - \mathbf{A}^{(2)})^{-1}[\kappa_2\underline{b}^{\otimes 2} + (\underline{b} \otimes \mathbf{A}_{(1)} + \mathbf{A}_{(1)} \otimes \underline{b})\underline{\mu}_1^c]$ where $\mathbf{A}_{(1)} = E\{\xi_t\mathbf{A}_t|\Im_{\mathbf{A}}\}$. Secondly, for any $\tau \geqslant 1$ we have the Yule–Walker type equation

$$\begin{aligned}\mu_2^c(\tau) &= \mathbf{A}\mu_2^c(\tau-1) + \underline{b}(\underline{\mu}_1^c)' = \left(\sum_{i=0}^{\tau-1}\mathbf{A}^i\right)\underline{b} \otimes (\underline{\mu}_1^c)' + \mathbf{A}^\tau\mu_2^c(0) \\ &= (I_{(p)} - \mathbf{A})^{-1}(I_{(p)} - \mathbf{A}^\tau)\underline{b} \otimes (\underline{\mu}_1^c)' + \mathbf{A}^\tau\mu_2^c(0), \quad \tau \geqslant 1.\end{aligned}$$

Then, because $\mu^c(\tau) = (\mu_2^c(\tau - i + j))_{1<l,j\leqslant p}$ with $\mu_2^c(\tau) = \mu_2^c(-\tau) = E(Y_tY_{t-\tau}|\Im_{\mathbf{A}})$, the covariance function $(\gamma(\tau))_{\tau\geqslant 0}$ of the model (2) can be easily recovered from the conditional covariance function of its vectorial representation by means of an appropriately selection matrix $C'(E\{\mu^c(\tau)\} - (E\{\underline{\mu}_1^c\})(E\{\underline{\mu}_1^c\})')C$.

Remark 2. When $p = 1$, the previous computations give $\mu_1^c = \frac{A_0}{1-A_1}$, $\mu_2^c(\tau) = \frac{A_1^\tau A_0^2(\kappa_2-1)}{(1-A_1)^2(1-\kappa_2A_1^2)} + \frac{A_0^2}{(1-A_1)^2}$ and $\gamma(\tau) = E\{\frac{A_1^\tau A_0^2(\kappa_2-1)}{(1-A_1)^2(1-A_1^2\kappa_2)}\} + Var\{\frac{A_0}{1-A_1}\}$ which reduce to standard expression in a nonrandom environment. Hence, the covariance function of $(Y_t)_{t\in\mathbb{Z}}$ is nonsummable.

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