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Dynamical Systems

Maslov index for solitary waves obtained as a limit of the Maslov index for periodic waves

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Abstract

A Maslov index for a solitary wave can be defined by approximating the solitary wave with periodic waves: when a sequence of periodic waves ϕ^α converges to the solitary wave ϕ, then the sequence of Maslov indices converges and its limit can be used as a definition for the Maslov index of ϕ. ***To cite this article: F. Chardard, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Un indice de Maslov pour les ondes solitaires obtenu comme limite de l'indice de Maslov pour les ondes périodiques. On peut définir l'indice de Maslov pour une onde solitaire en approchant l'onde solitaire par des ondes périodiques : lorsqu'une suite d'ondes périodiques ϕ^α converge vers l'onde solitaire ϕ, alors sa limite peut-être utilisée comme définition de l'indice de Maslov de ϕ. ***Pour citer cet article : F. Chardard, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

L'indice de Maslov est un entier attribué à des orbites de systèmes hamiltoniens. Il est utilisé dans de nombreux domaines comme la mécanique quantique [2] ou la détermination de la stabilité des ondes solitaires [8,3].

C'est ce dernier domaine que vise cette Note. Pour certaines équations aux dérivées partielles 1D, une onde ϕ se propageant sans se déformer vérifie une équation hamiltonienne de type :

$$\mathbf{J}\mathbf{u}_x = \nabla_{\mathbf{u}} H, \quad H : \mathbb{R}^{2n} \to \mathbb{R} \tag{1}$$

La recherche des valeurs propres de la linéarisation d'une EDP autour d'une onde ϕ, ou de la hessienne du Hamiltonien d'une EDP, conduit à résoudre des problèmes du type :

$$\mathbf{J}\mathbf{z}_x = \mathbf{C}(x,\lambda)\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^{2n},\ \lambda \in \mathbb{R}, \tag{2}$$

avec $\mathbf{C}(x,\lambda)$ symétrique et $\lim_{|x|\to\infty} -\mathbf{J}\mathbf{C}(x,\lambda) = \mathbf{B}_\infty(\lambda)$ et $\mathbf{C}(x,0) = D^2 H_{\phi(x)}$.

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Le comptage des solutions bornées de ce type de systèmes conduit à considérer la variété Λ_n des espaces lagrangiens (Un espace $\mathcal{V}$ est dit lagrangien s'il est de dimension n et si $\forall(u,v)\in\mathcal{V}^2\ {}^t u\mathbf{J}v=0$), dont le groupe fondamental est $\mathbb{Z}$. On peut donc associer un entier à tout chemin fermé de cette variété, appelé indice de Maslov.

Dans le cas où ϕ est une onde périodique, il est possible de définir un indice de Maslov en λ dans les cas suivants :

– Si pour un λ donné, le système n'a pas de solution bornée, alors l'espace des solutions tendant vers 0 en $-\infty$ forme un chemin fermé dans Λ_n sur une période. L'indice de Maslov de l'onde en λ est alors défini comme l'indice de Maslov de ce chemin et noté $I_{\text{per}}(\phi,\lambda)$.
– Si $\lambda=0$ et si l'espace des solutions $\mathcal{R}$ de (2) tendant vers 0 en $-\infty$ est de dimension $n-1$, alors $\mathbb{R}\phi_x\oplus\mathcal{R}$ forme un chemin fermé dans Λ_n sur une période. L'indice de Maslov de l'onde en $\lambda=0$ est alors défini comme l'indice de Maslov de ce chemin, et noté $I_{\text{per}}(\phi)$.

On cherche à étendre ces définitions aux ondes solitaires. On approche donc une onde solitaire ϕ par des ondes périodiques ϕ^α, de période $\frac{2\pi}{\alpha}$. On peut alors faire tendre α vers 0 et obtenir ainsi un indice de Maslov pour ϕ pour différentes valeurs de λ. On définit $h(\alpha)=H(\phi^\alpha)$ et on suppose que $\frac{\partial\mathbf{C}}{\partial\lambda}$ est une matrice symétrique positive. Moyennant certaines hypothèses, on peut alors prouver que les limites de $I_{\text{per}}(\phi^\alpha,\lambda)$ et $I_{\text{per}}(\phi^\alpha)$ lorsque $\alpha\to 0$ sont respectivement :

– l'indice de Maslov de l'espace des solutions de (2) tendant vers 0 en $-\infty$ lorsqu'il forme un lacet sur $\mathbb{R}$. On note alors $I_{\text{hom}}(\lambda)$ cette quantité.
– $\begin{cases}\lim_{\lambda\to 0^+} I_{\text{hom}}(\lambda) & \text{si } h'<0,\\ \lim_{\lambda\to 0^-} I_{\text{hom}}(\lambda) & \text{si } h'>0\end{cases}$ dans le cas où l'ensemble des solutions $L^2(\mathbb{R})$ est réduit à $\mathbb{R}\phi_x$.

On obtient ainsi un indice de Maslov pour les ondes solitaires. Des applications de cette construction seront rapportées dans [4].

1. Introduction

The Maslov index is an integer associated to orbits of Hamiltonian systems that is used in a wide range of physical applications: semi-classical quantization, quantum chaology, classical mechanics [2], etc. It is also used as a counter of eigenvalues for some self-adjoint operators in [8,3] to determine the stability of travel ling waves of Schrödinger or Fitzhugh–Nagumo equations. There, the Maslov index is defined directly on the solitary waves using the intersection number of paths in the Lagrangian Grassmannian.

Unfortunately determining intersections can be uneasy. Therefore, an other definition for the Maslov index is desirable.

In this Note a new definition of Maslov index for a solitary wave is given, when the solitary wave is obtained as a limit of periodic waves, by using the classical definition of Maslov index for periodic orbits, taking limits, and proving that the index converges.

A 1D traveling wave ϕ is often a solution of an autonomous non-linear Hamiltonian system:

$$\mathbf{J}\mathbf{u}_x=\nabla_{\mathbf{u}}H,\quad H:\mathbb{R}^{2n}\to\mathbb{R}. \tag{3}$$

Stability of traveling waves has been linked to the spectrum of the second variation of the Hamiltonian (see [6,4]). Tracking the eigenvalues λ of the linearization of some PDE, or of the second variation of a Hamiltonian of a PDE near a traveling wave can lead to a Hamiltonian system in the form:

$$\mathbf{z}_x=\mathbf{B}(x,\lambda)\mathbf{z},\quad \mathbf{B}(x,\lambda)=-\mathbf{J}\mathbf{C}(x,\lambda),\ \mathbf{J}=\begin{pmatrix}\mathbf{0} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix},\ \mathbf{z}\in\mathbb{R}^{2n} \tag{4}$$

where $\mathbf{C}(x,\lambda)$ is a symmetric matrix which usually satisfies:

$$\mathbf{C}(x,0)=D^2H_{\phi(x)}. \tag{5}$$

The latter formula means that, for $\lambda=0$, system (4) is the linearization of system (3).

An n-dimensional subspace, $\operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subset \mathbb{R}^{2n}$, is a Lagrangian subspace if $\langle \mathbf{J}\mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for each $\mathbf{u}_i, \mathbf{u}_j$. The set of all Lagrangian subspaces is a compact manifold of dimension $\frac{1}{2}n(n+1)$ and is denoted by Λ_n. A point $\mathcal{W}$ in Λ_n can be represented by a $2n \times n$ matrix $\mathbf{W}$, where $\mathbf{W}$ is of rank n and satisfies $\mathbf{W}^T\mathbf{J}\mathbf{W} = 0$. Lagrangian subspaces are invariant under the flow of (4). Hence, $\mathbf{W}(x, \lambda)$ satisfying

$$\mathbf{J}\mathbf{W}_x = \mathbf{C}(x, \lambda)\mathbf{W}, \quad \mathbf{W}(x_0, \lambda)^T\mathbf{J}\mathbf{W}(x_0, \lambda) = 0, \quad \mathbf{W}(x_0, \lambda) \text{ has rank } n, \quad x \geqslant x_0, \tag{6}$$

defines a path of Lagrangian subspaces, since $\frac{d}{dx}\mathbf{W}(x, \lambda)^T\mathbf{J}\mathbf{W}(x, \lambda) = 0$ and so

$$\mathbf{W}(x, \lambda)^T\mathbf{J}\mathbf{W}(x, \lambda) = \mathbf{W}(x_0, \lambda)^T\mathbf{J}\mathbf{W}(x_0, \lambda) = 0.$$

Given a path of Lagrangian subspaces, define the *Maslov angle* $\kappa(x, \lambda)$ by

$$\mathrm{e}^{\mathrm{i}\kappa(x,\lambda)} = s\big(\mathcal{W}(\cdot, \lambda)\big) = \frac{\det(\mathbf{D}_1(x, \lambda) - \mathrm{i}\mathbf{D}_2(x, \lambda))}{\det(\mathbf{D}_1(x, \lambda) + \mathrm{i}\mathbf{D}_2(x, \lambda))}, \quad \mathbf{W}(x, \lambda) = \begin{pmatrix} \mathbf{D}_1(x, \lambda) \\ \mathbf{D}_2(x, \lambda) \end{pmatrix}. \tag{7}$$

It can be noticed that $s(\mathbf{AP}) = s(\mathbf{A})$ for any invertible $n \times n$-matrix $\mathbf{P}$ and therefore s is independent of the choice of $\mathbf{W}$ among representatives of $\mathcal{W}$. Moreover s is a continuous function over Λ_n (see [2]).

When this path is closed in Λ_n, i.e. when there exists a $n \times n$ matrix $\mathbf{P}$ such that $\mathbf{W}(x_0, \lambda) = \mathbf{W}(x_1, \lambda)\mathbf{P}$, then the *Maslov index* is an integer defined by

$$m\big(\mathcal{W}(\cdot, \lambda)\big) = \frac{\kappa(x_1, \lambda) - \kappa(x_0, \lambda)}{2\pi}. \tag{8}$$

This index m is well-defined for any closed path in Λ_n and gives a one-to-one correspondence between $\mathbb{Z}$ and homotopy classes of closed paths in Λ_n (see [2]).

Defining a Maslov index for a linear Hamiltonian system can be done by choosing a Lagrangian plane of solutions and an x-interval on which it is closed.

Definition 1. The unstable space $\mathcal{U}(\cdot, \lambda)$ of system (4) is defined as the set of solutions of (4) which decay to 0 exponentially at $-\infty$ (its dimension may vary with λ).

2. Maslov index for hyperbolic periodic waves

Suppose that the studied wave is *L-periodic*. Then, $\mathbf{C}(x, \lambda)$ is L-periodic with respect to x. When $\dim(\mathcal{U}(\cdot, \lambda)) \in \{n-1, n\}$, there are two cases when a Maslov index can be defined [5]:

- The unstable space $\mathcal{U}(\cdot, \lambda)$ is a n-dimensional space. The periodic system is said to be strictly hyperbolic.

Definition 2. When $\mathcal{U}(\cdot, \lambda)$ is a n-dimensional space, the Maslov index at λ of system (4) is defined as $m(\mathcal{U}(\cdot, \lambda))$, where $\mathcal{U}(\cdot, \lambda)$ is taken over $[0, L]$.

- The L-periodic traveling wave ϕ is a solution of (3) and the linear system (4) at $\lambda = 0$ is the linearization of system (3). ϕ_x is then a solution of the linear system at $\lambda = 0$ and therefore, $\mathcal{U}(\cdot, 0)$ is not n-dimensional. However, if it is $n-1$-dimensional, the following definition can be used:

Definition 3. If the dimension of the space $\mathcal{R}(x) = \mathcal{U}(x, 0)$ of solutions of $\mathbf{J}\mathbf{z}_x = D^2H_\phi\mathbf{z}$ decaying to 0 at $-\infty$ is $n-1$, then $(\mathbb{R}\phi_x \oplus \mathcal{R})_{|[0,L]}$ is a closed path over one period in the Lagrangian manifold and the Maslov index of ϕ at 0 is defined as $I_{\mathrm{per}}(\phi) = m((\mathbb{R}\phi_x \oplus \mathcal{R})_{|[0,L]})$.

3. General hypotheses made on the solitary wave

We suppose now that ϕ is a *solitary* wave. Naturally, the two previous definitions will require a separate treatment in order to be extended. However, these two cases share common hypotheses:

Hypothesis 4.

- $\mathbf{B}(x, \lambda)$ is a smooth function with respect to x and analytic with respect to λ.

– There exists $\mathbf{B}_\infty(\lambda)$, $\gamma > 0$ and $F > 0$ such that $\forall x, \lambda$ $\|\mathbf{B}(x,\lambda) - \mathbf{B}_\infty(\lambda)\| \leqslant F\mathrm{e}^{-\gamma|x|}$.
– There is an open set $\mathbb{X}$ of real numbers, such that for each $\lambda \in \mathbb{X}$, $Sp(\mathbf{B}_\infty(\lambda)) \cap \mathrm{i}\mathbb{R} = \emptyset$.
– The set $\sigma_p = \{\lambda \in \mathbb{X} \mid \text{system (4) admits a non-trivial bounded solution}\}$ is a strict subset of $\mathbb{X}$.

Then we can write: $\mathbb{R}^{2n} = E_u(\mathbf{B}_\infty(\lambda)) \oplus E_s(\mathbf{B}_\infty(\lambda))$ where $E_u(\mathbf{B}_\infty(\lambda))$ (resp. $E_s(\mathbf{B}_\infty(\lambda))$) is the unstable (resp. stable) space of $\mathbf{B}_\infty(\lambda)$, i.e. sum of generalized eigenspaces associated to eigenvalues with strictly positive (resp. negative) real part. Since $\mathbf{J}\mathbf{B}_\infty(\lambda)$ is symmetric, $\dim E_u(\mathbf{B}_\infty(\lambda)) = \dim E_s(\mathbf{B}_\infty(\lambda))$. Therefore $E_u(\mathbf{B}_\infty(\lambda))$ and $E_s(\mathbf{B}_\infty(\lambda))$ have the same dimension: n.

Moreover, $\mathcal{U}(x,\lambda)$ has dimension n, is Lagrangian and $\lim_{x\to-\infty}\mathcal{U}(x,\lambda) = E_u(\mathbf{B}_\infty(\lambda))$. Symmetrically, the set of solutions that decay as $x \to +\infty$, which is called the stable space $\mathcal{S}(x,\lambda)$, which is Lagrangian and $\lim_{x\to+\infty}\mathcal{S}(x,\lambda) = E_s(\mathbf{B}_\infty(\lambda))$.

As we wish to define the Maslov index as a limit of the periodic case, we will also suppose that there is a family of periodic waves ϕ^α which approaches the solitary wave ϕ. More precisely, consider a family of systems parametrized by $\alpha \in [0, \alpha_0]$:

$$\mathbf{J}\mathbf{z}_x = \mathbf{C}^\alpha(x,\lambda)\mathbf{z}. \tag{9}$$

We suppose that:

Hypothesis 5.

– $\mathbf{C}^\alpha(x,\lambda)$ is a smooth function with respect to x and α, and analytic with respect to λ and $\mathbf{C}^0(x,\lambda) = \mathbf{C}(x,\lambda)$.
– $\mathbf{C}^\alpha(\cdot,\lambda)$ is L_α-periodic when $\alpha > 0$ and $\lim_{\alpha\to0^+} L_\alpha = +\infty$.
– $\forall M > 0$ $\lim_{\alpha\to0}\sup_{x\in[-\frac{L_\alpha}{2},\frac{L_\alpha}{2}],\lambda\in\mathbb{X},|\lambda|<M}\|\mathbf{C}^\alpha(x,\lambda) - \mathbf{C}(x,\lambda)\| = 0$.

4. Maslov index for solitary waves when $\lambda \in \mathbb{X} - \sigma_p$

In this section, we will extend Definition 2 and suppose that $\lambda \in \mathbb{X} - \sigma_p$.

By hypothesis, the space of solutions decaying both at $-\infty$ and $+\infty$ is reduced to $\{0\}$.

Then $\lim_{x\to+\infty}\mathcal{U}(x,\lambda) = E_u(\mathbf{B}_\infty(\lambda))$ (Lemma 3.7 in [1]).

Therefore, $x \mapsto \mathcal{U}(x,\lambda)$ is a closed path in Λ_n over $\mathbb{R}$ for each $\lambda \in \mathbb{X} - \sigma_p$.

The quantity $m(\mathcal{U}(\cdot,\lambda))$ is therefore well defined. Let us now show that it is the limit[1] of $m(\mathcal{U}^\alpha(\cdot,\lambda))$ when $\alpha \to 0$.

Let $d(\cdot,\cdot)$ be a metric on Λ_n compatible with its compact manifold structure (it can be obtained by embedding Λ_n in $\mathbb{R}^N$).

Now, use Lemma 2.11, p. 172 of Gardner [7]:

Theorem 6 *(Gardner). If Hypotheses* 4, 5 *are satisfied, let $\lambda \in \mathbb{X} - \sigma_p$. Then, for α small enough, the space $\mathcal{U}^\alpha(\cdot,\lambda)$ of solutions of* (9) *decaying at $-\infty$ is n-dimensional and converges x-uniformly to $\mathcal{U}(\cdot,\lambda)$:*

$$\lim_{\alpha\to0}\sup_{x\in[-\frac{L_\alpha}{2},\frac{L_\alpha}{2}]} d\big(\mathcal{U}^\alpha(x,\lambda),\mathcal{U}(x,\lambda)\big) = 0. \tag{10}$$

Corollary 4.1. *Let $\lambda \in \mathbb{X} - \sigma_p$. For α small, $I_{\mathrm{per}}(\phi^\alpha,\lambda)$ is well defined and equal to $m(\mathcal{U}^\alpha(\cdot,\lambda))$.*

Corollary 4.2. $\lim_{\alpha\to0}\sup_{x\notin[-\frac{L_\alpha}{2},\frac{L_\alpha}{2}]} d(\mathcal{U}^\alpha(\frac{L_\alpha}{2},\lambda),\mathcal{U}(x,\lambda)) = 0$.

Proof. $d(\mathcal{U}^\alpha(\frac{L_\alpha}{2},\lambda),\mathcal{U}(x,\lambda)) \leqslant d(\mathcal{U}^\alpha(\frac{L_\alpha}{2},\lambda),\mathcal{U}(\frac{L_\alpha}{2},\lambda)) + d(\mathcal{U}(\frac{L_\alpha}{2},\lambda), E_u(\mathbf{B}_\infty(\lambda))) + d(E_u(\mathbf{B}_\infty(\lambda)),\mathcal{U}(x,\lambda))$.

Using $\lim_{|x|\to\infty}\mathcal{U}(x,\lambda) = E_u(\mathbf{B}_\infty(\lambda))$, Corollary 4.2 is immediate.

Now define: $\mathcal{V}^\alpha(x,\lambda) = \begin{cases}\mathcal{U}^\alpha(x,\lambda) & \text{if } x \in [-\frac{L_\alpha}{2},\frac{L_\alpha}{2}],\\ \mathcal{U}^\alpha(\frac{L_\alpha}{2},\lambda) & \text{if } x \notin [-\frac{L_\alpha}{2},\frac{L_\alpha}{2}].\end{cases}$

[1] Of course, as m is an integer-valued function, it means that $m(\mathcal{U}^\alpha(\cdot,\lambda))$ is equal to $m(\mathcal{U}(\cdot,\lambda))$ for small enough α.

From (10) and Corollary 4.2, we get: $\lim_{\alpha\to 0}\sup_{x\in\mathbb{R}} d(\mathcal{V}^\alpha(x,\lambda),\mathcal{U}(x,\lambda))=0$.
Λ_n is compact and therefore s is uniformly continuous over it and consequently:

$$\lim_{\alpha\to 0}\sup_{x\in\mathbb{R}}\left|s\big(\mathcal{V}^\alpha(x,\lambda)\big)-s\big(\mathcal{U}(x,\lambda)\big)\right|=0.$$

Hence, if κ and κ^α are angles associated to $\mathcal{U}$ and $\mathcal{V}^\alpha$, then

$$\lim_{\alpha\to 0}\sup_{x\in\mathbb{R}}\left|\kappa^\alpha(x,\lambda)-\kappa^\alpha(0)-\kappa(x,\lambda)+\kappa(0)\right|=0.$$

Therefore:

$$\begin{aligned}\lim_{\alpha\to 0}\left|m\big(\mathcal{U}(\cdot,\lambda)\big)-m\big(\mathcal{V}^\alpha(\cdot,\lambda)\big)\right| &= \lim_{\alpha\to 0}\left|\lim_{x\to\infty}\kappa(x,\lambda)-\lim_{x\to-\infty}\kappa(x,\lambda)-\lim_{x\to\infty}\kappa^\alpha(x,\lambda)+\lim_{x\to\infty}\kappa^\alpha(x,\lambda)\right|\\ &=0. \qquad\square\end{aligned}$$

This proves that the limit of $I_{\text{per}}(\phi^\alpha,\lambda)=m(\mathcal{V}^\alpha(\cdot,\lambda))=m(\mathcal{U}^\alpha(\cdot,\lambda))$ as $\alpha\to 0$ exists. This limit is the basis for our definition of Maslov index:

Definition 7. The Maslov index of system (4) for $\lambda\in\mathbb{X}-\sigma_p$ is *defined* as $I_{\text{hom}}(\phi,\lambda)=m(\mathcal{U}(\cdot,\lambda))$.

5. Defining a Maslov index for solitary waves when the system is also the linearization of an autonomous system ($\boldsymbol{\lambda=0}$)

Using Eqs. (3)–(5), it is easy to see that ϕ_x is a solution of system (4) at $\lambda=0$ and therefore $0\in\sigma_p$. To extend Definition 3, make the following additional hypothesis:

Hypothesis 8.

- There is a unique family of L_α-periodic waves ϕ^α solutions of (3) such that ϕ^α converges to ϕ and $\mathbf{C}^\alpha(x,0)=D^2H_{\phi^\alpha(x)}$.
- The functions $h(\alpha)=H(\phi^\alpha)$ and $l(\alpha)=L_\alpha$ are differentiable and in $]0,\alpha_0[$, we have $h'\neq 0$, $l'<0$.
- $\partial_\lambda\mathbf{C}^\alpha(\cdot,\lambda)$ is positive[2] in the sense of symmetric matrices.
- $0\in\mathbb{X}$ and the space of bounded solutions of the linear system (4) at $\lambda=0$ is equal to $\mathbb{R}\phi_x$.

Let $\mathbf{M}^\alpha(\lambda)$ be the matrix such that $\mathbf{z}(L_\alpha)=\mathbf{M}^\alpha(\lambda)\mathbf{z}(0)$ for any $\mathbf{z}$ solution of $\mathbf{J}\mathbf{z}_x=\mathbf{C}^\alpha(x,\lambda)\mathbf{z}$.

For small enough α, $\mathbf{M}^\alpha(0)$ has only two eigenvalues at $+1$, the others being off the unit circle. Therefore, $\mathcal{R}^\alpha=\mathcal{U}^\alpha(\cdot,0)$ has dimension $n-1$ and $I_{\text{per}}(\phi^\alpha)$ is well-defined.

Let first assume that $h'(\alpha)>0$. Then there exists a basis (x,y) of $E_1(\mathbf{M}^\alpha(0))$ and $\gamma>0$ such that ${}^t yJx=1$ and $\left(\begin{smallmatrix}1&\gamma\\0&1\end{smallmatrix}\right)$ is the matrix of $\mathbf{M}^\alpha(0)_{|E_1(\mathbf{M}^\alpha(0))}$ in (x,y).

For λ near 0^+, there is one pair of eigenvalues of $\mathbf{M}^\alpha(\lambda)$ on the unit circle, the upper eigenvalue having a positive Krein sign.[3] For λ near 0^-, all the eigenvalues of $\mathbf{M}^\alpha(0)$ are off the unit circle, the unstable space $\mathcal{U}^\alpha(\cdot,\lambda)$ has dimension n and $\lim_{\lambda\to 0^-}\mathcal{U}^\alpha(\cdot,\lambda)=\mathbb{R}\phi_x^\alpha\oplus\mathcal{R}^\alpha$, x-uniformly.

If $h'(\alpha)<0$, then the sign of γ and the Krein sign are reversed, and the two critical eigenvalues are on the unit circle when λ is close to 0^- and $\lim_{\lambda\to 0^+}\mathcal{U}^\alpha(\cdot,\lambda)=\mathbb{R}\phi_x^\alpha\oplus\mathcal{R}^\alpha$, x-uniformly (see Fig. 1).

Therefore, for small enough α, the Maslov index of ϕ^α is $I_{\text{per}}(\phi^\alpha)=\begin{cases}\lim_{\lambda\to 0^-}I_{\text{per}}(\phi^\alpha,\lambda) & \text{if } h'(\alpha)>0,\\ \lim_{\lambda\to 0^+}I_{\text{per}}(\phi^\alpha,\lambda) & \text{if } h'(\alpha)<0.\end{cases}$

Therefore, we *define* the Maslov index of the soliton ϕ as $I_{\text{hom}}(\phi)=\begin{cases}\lim_{\lambda\to 0^-}I_{\text{hom}}(\phi,\lambda) & \text{if } h'_{|]0,\alpha_0[}>0,\\ \lim_{\lambda\to 0^+}I_{\text{hom}}(\phi,\lambda) & \text{if } h'_{|]0,\alpha_0[}<0.\end{cases}$

[2] The case $\partial_\lambda\mathbf{C}^\alpha(\cdot,\lambda)$ negative can be handled similarly, by replacing λ by $-\lambda$.

[3] Suppose $\mathrm{e}^{\mathrm{i}\alpha}$, with $0<\alpha<\pi$, is a simple eigenvalue of $\mathbf{M}^\alpha(\lambda)$ with eigenvector ξ, then the Krein sign of $\mathrm{e}^{\mathrm{i}\alpha}$ is

$$\operatorname{sign}(-\mathrm{i}\xi^*\mathbf{J}\xi),\quad \text{where } * \text{ indicates complex conjugate transpose.}$$

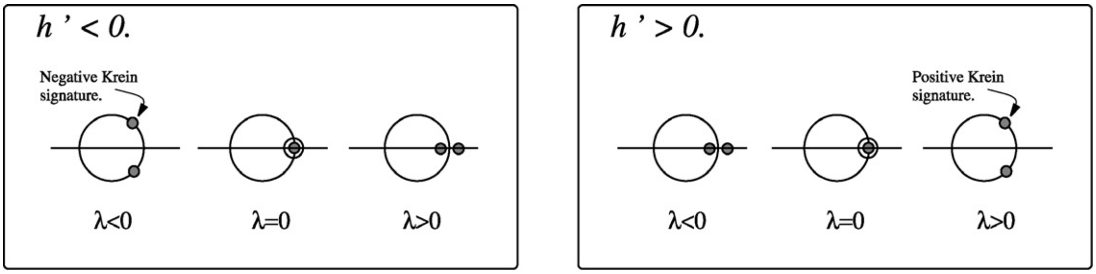

Fig. 1. Position of the two critical eigenvalues of $\mathbf{M}^{\alpha}(\lambda)$ in the case where λ is close to 0 and $\partial_\lambda \mathbf{C}^{\alpha}(\cdot,\cdot)$ is positive.

Fig. 1. Position des deux valeurs propres critiques de $\mathbf{M}^{\alpha}(\lambda)$ lorsque λ est proche de 0 et que $\partial_\lambda \mathbf{C}^{\alpha}(\cdot,\cdot)$ est positif.

6. Conclusion

A natural extension of the Maslov index when $\lambda \in \mathbb{X} - \sigma_p$ has been provided. Besides, by defining a Maslov index at $\lambda = 0$, the pure ODE problem (3) has been linked to the spectral problem (4). This index is of interest for the stability analysis of waves (see [4] for applications).

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