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Partial Differential Equations

The theory of De Giorgi for non-local operators

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Abstract

Under quite general assumptions on $k:\mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, we study integro-differential operators $\mathscr{L}$ of the form

$$(\mathscr{L}u)(x) = 2 \lim_{\varepsilon\to 0} \int\limits_{\substack{y\in\mathbb{R}^d\\ |y-x|>\varepsilon}} \big(u(y)-u(x)\big)k(x,y)\,\mathrm{d}y. \tag{1}$$

Our assumptions on k imply that there is $\alpha \in (0,2)$ such that $k(x,y)|x-y|^{d+\alpha}$ stays bounded for small $|x-y|$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Set $D_\Omega(\mathscr{E}) = L^\infty(\mathbb{R}^d) \cap H^{\alpha/2}_{\mathrm{loc}}(\Omega)$. We call a function $u \in D_\Omega(\mathscr{E})$ *$\mathscr{L}$-harmonic in Ω* if for any $\phi \in C_0^\infty(\Omega)$

$$\mathscr{E}(u,\phi) = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(y)-u(x)\big)\big(\phi(y)-\phi(x)\big)k(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2}$$

The aim of this Note is to prove local bounds for $\mathscr{L}$-harmonic functions. The main results says that functions $u \in D_B(\mathscr{E})$ which are $\mathscr{L}$-harmonic in the ball B satisfy a priori estimates in $C^\beta(\overline{B'})$ for some $\beta > 0$ and any $B' \Subset B$. The results can be seen as a generalization of the so-called De Giorgi–Nash–Moser theory to integro-differential operators of order $\alpha \in (0,2)$. ***To cite this article: M. Kassmann, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

La théorie de De Giorgi pour les opérateurs non locaux. Sous des conditions générales pour $k:\mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, nous étudions les opérateurs intégro-différentiels $\mathscr{L}$ de type (1). Nos conditions pour k impliquent qu'il existe un $\alpha \in (0,2)$ tel que $k(x,y)|x-y|^{d+\alpha}$ reste borné pour de petits $|x-y|$. Soit $\Omega \subset \mathbb{R}^d$ un ouvert borné. Soit $D_\Omega(\mathscr{E}) = L^\infty(\mathbb{R}^d) \cap H^{\alpha/2}_{\mathrm{loc}}(\Omega)$. Une fonction $u \in D_\Omega(\mathscr{E})$ est nommée *$\mathscr{L}$-harmonique en Ω* si pour tout $\phi \in C_0^\infty(\Omega)$ $\mathscr{E}(u,\phi) = 0$. Le but de cette Note est de trouver des bornes locales pour des fonctions $\mathscr{L}$-harmoniques. Les principaux resultats démontrent que des fonctions $u \in D_B(\mathscr{E})$ qui sont $\mathscr{L}$-harmoniques dans la boule B satisfont des estimations a priori dans $C^\beta(\overline{B'})$ pour un $\beta > 0$ et pour tout $B' \Subset B$. Les résultats de ce travail peuvent être regardés comme une généralisation de la théorie dite De Giorgi–Nash–Moser aux opérateurs integro-differentiels d'ordre $\alpha \in (0,2)$. ***Pour citer cet article : M. Kassmann, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

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1. Introduction

Our aim is to apply the method of De Giorgi to non-local operators and to answer the question of local regularity for non-local Dirichlet forms. This is the natural next step after boundedness of resolvents has been shown in [3]. Let $k:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$, $d\geqslant 2$, be measurable and satisfy

$$k(x,y)=k(y,x)\quad\text{for almost all } x,y\in\mathbb{R}^d, \tag{3}$$

$$\nu\leqslant k(x,y)|x-y|^{d+\alpha}\leqslant\nu^{-1}\quad\text{for almost all } x,y\in\mathbb{R}^d \text{ with } |x-y|\leqslant 1, \tag{4}$$

$$k(x,y)\leqslant M|x-y|^{-d-\eta}\quad\text{for almost all } x,y\in\mathbb{R}^d \text{ with } |x-y|>1, \tag{5}$$

for some $\alpha\in(0,2)$, $\nu\in(0,1)$, $\eta>0$, $M\geqslant 1$. The assumptions could be relaxed further. We introduce a method for proving local estimates of functions $u:\mathbb{R}^d\to\mathbb{R}$ satisfying $(\mathscr{L}u)(x)=0$ for $x\in\Omega$, where $\mathscr{L}$ is as in (1) and $\Omega\subset\mathbb{R}^d$ is a bounded open set. For $u,v\in H^{\alpha/2}(\mathbb{R}^d)$ recall the definition of $\mathscr{E}(u,v)$ from (2). Set $D(\mathscr{E})=\{u\in L^2(\mathbb{R}^d):\ \mathscr{E}(u,u)<\infty\}$. Under Assumptions (3)–(5), $D(\mathscr{E})=H^{\alpha/2}(\mathbb{R}^d)$. The tuple $(\mathscr{E},D(\mathscr{E}))$ is a regular Dirichlet form.

Definition 1.1. A function $u\in D_\Omega(\mathscr{E})=L^\infty(\mathbb{R}^d)\cap H^{\alpha/2}_{\mathrm{loc}}(\Omega)$ is called $\mathscr{L}$-subharmonic in Ω if $\mathscr{E}(u,\phi)\leqslant 0$ for all test functions $\phi\in C_0^\infty(\Omega)$, $\phi\geqslant 0$. A function $u\in D_\Omega(\mathscr{E})$ is $\mathscr{L}$-superharmonic in Ω if $-u$ is $\mathscr{L}$-subharmonic in Ω. A function $u\in D_\Omega(\mathscr{E})$ is $\mathscr{L}$-harmonic in Ω if u and $-u$ are $\mathscr{L}$-subharmonic in Ω, which is equivalent to

$$\mathscr{E}(u,\phi)=0\quad\forall\phi\in C_0^\infty(\Omega). \tag{6}$$

Note that $u\in D_\Omega(\mathscr{E})$ implies finiteness of $\mathscr{E}(u,\phi)$ for any $\phi\in C_0^\infty(\Omega)$. Our aim is to sketch the ideas of how our main result, Theorem 2.6, can be proved. Full details are worked out in [4] and will be published elsewhere. The following auxiliary result on real numbers can be used as a substitute for integration by parts:

Lemma 1.2. *Let $a,b>0$, $p>1$ and $\tau_1,\tau_2\geqslant 0$. Then*

$$(b-a)\big(\tau_1^{p+1}a^{-p}-\tau_2^{p+1}b^{-p}\big)\geqslant\frac{\tau_1\tau_2}{p-1}\left(\left(\frac{b}{\tau_2}\right)^{\frac{-p+1}{2}}-\left(\frac{a}{\tau_1}\right)^{\frac{-p+1}{2}}\right)^2 - \max\left\{4,\frac{6p-5}{2}\right\}(\tau_2-\tau_1)^2\left(\left(\frac{b}{\tau_2}\right)^{-p+1}+\left(\frac{a}{\tau_1}\right)^{-p+1}\right).$$

2. Results

Our main result is Theorem 2.6. In order to prove it, we provide estimates of $\inf_{x\in B_R}u(x)$ from below for non-negative $\mathscr{L}$-superharmonic functions and prove a priori estimates in Hölder spaces. The following result is a version of a classical tool used in regularity theory. This tool can be found in many sources, see [2,7,6]. Our result differs from the classical ones since there is one additional assumption, inequality (9). This assumption takes care of non-local terms. It is interesting that, despite this additional assumption, Theorem 2.1 still implies Hölder regularity.

Theorem 2.1. *There are $\kappa>0$ and $\gamma\in(0,1)$ such for any $x_0\in\mathbb{R}^d$, $R\in(0,1/8)$, $u\in D_{B_{2R}(x_0)}(\mathscr{E})$ satisfying*

$$u \text{ is } \mathscr{L}\text{-superharmonic and non-negative in } B_{2R}(x_0), \tag{7}$$

$$\big|\{x\in B_R(x_0):\ u(x)\geqslant 1\}\big|\geqslant\tfrac12\big|B_R(x_0)\big|, \tag{8}$$

$$u(x)-1\geqslant 1-2\left(4\frac{|x-x_0|}{R}\right)^\gamma\quad\text{for almost any } x\in\mathbb{R}^d\setminus B_R(x_0), \tag{9}$$

the following holds: $\operatorname{ess-inf}_{B_{\frac{R}{4}}(x_0)}u\geqslant\kappa$. *The constants γ,κ depend only on d and on the constants appearing in assumptions* (3)–(5).

We need several auxiliary results before we can prove Theorem 2.1. One key argument in the proof of Theorem 2.1 is the following Morrey–Besov-type inequality for $\log u$:

Lemma 2.2. *For $z_0 \in \mathbb{R}^d$, $s > 0$, write B_s instead of $B_s(z_0)$. Assume $r > 0$, $\rho \in (0, r]$ satisfy $2r + 4\rho \leqslant 1$. Assume $u \in D_{B_{2r}}(\mathscr{E})$ is $\mathscr{L}$-superharmonic in B_{2r}, $u \geqslant \delta > 0$ a.e. in B_{2r}, and*

$$\int_{\mathbb{R}^d \setminus B_{r+\rho}} u(x)k(x, y)\,\mathrm{d}x \geqslant 0 \quad \textit{for almost any } y \in B_{r+\rho}. \tag{10}$$

Then

$$\iint_{B_r \times B_r} \left(\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right)|x - y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x \leqslant c\rho^{-\alpha}|B_{r+\rho}| \tag{11}$$

where $c > 0$ is independent of u, z_0, r, ρ, and δ.

In order to grasp the implication of Lemma 2.2, one can set $\rho = r$. Inequality (11) implies $\log u \in \mathrm{BMO}$. The John-Nirenberg embedding then leads to the following result:

Lemma 2.3. *For $x_0 \in \mathbb{R}^d$, $s > 0$, write B_s instead of $B_s(x_0)$. Let $R \in (0, 1/8)$. Assume $u \in D_{B_{2R}}(\mathscr{E})$ is $\mathscr{L}$-superharmonic in B_{2R}, $u \geqslant \delta > 0$ a.e. in B_{2R}, and*

$$\int_{\mathbb{R}^d \setminus B_R} u(x)k(x, y)\,\mathrm{d}x \geqslant 0 \quad \textit{for almost any } y \in B_R. \tag{12}$$

Then there exist $\bar{p} \in (0, 1)$ and $c > 0$ such that

$$\left(\fint_{B_R} u(x)^{\bar{p}}\,\mathrm{d}x\right)^{1/\bar{p}}\mathrm{d}x \leqslant c\left(\fint_{B_R} u(x)^{-\bar{p}}\,\mathrm{d}x\right)^{-1/\bar{p}} \tag{13}$$

where c and $\bar{p}$ are independent of x_0, R, u, and δ.

Note that the constant c in (13) does not depend on values of u outside of $B_{2R}(x_0)$ although the operator $\mathscr{L}$ is non-local. The proof of Theorem 2.1 makes also use of Moser's iteration technique for non-negative exponents. The following lemma shows how a single iteration step can be obtained. The main technical ingredient is given by Lemma 1.2.

Lemma 2.4. *Assume $r > 0$, $\rho \in (0, r)$ satisfy $2r + 4\rho \leqslant 1$. For $x_0 \in \mathbb{R}^d$, $s > 0$, write B_s instead of $B_s(x_0)$. Let $p > 1$. Assume $u \in D_{B_{2r}}(\mathscr{E})$ is $\mathscr{L}$-superharmonic in B_{2r} and satisfies $u(x) \geqslant \delta$ for almost any $x \in B_{2r}$ and some $\delta > 0$. Then*

$$\iint_{B_r \times B_r} \frac{(u(y)^{\frac{-p+1}{2}} - u(x)^{\frac{-p+1}{2}})^2}{|x - y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \leqslant c \max\left\{\frac{p-1}{2}, \frac{6(p-1)^2}{16}\right\}\rho^{-\alpha}\int_{B_{r+\rho}} u(x)^{-p+1}\,\mathrm{d}x, \tag{14}$$

where $c > 0$ is independent of u, x_0, r, ρ, p, and δ.

As already mentioned, Lemma 2.4 can be iterated in order to estimate the infimum of a positive $\mathscr{L}$-superharmonic function. The iteration scheme is almost identical to the one in [8]. It results in the following corollary:

Corollary 2.5. *Assume $R \in (0, 1/8)$, $\mu \in (0, 1)$, $x_0 \in \mathbb{R}^d$. Assume $u \in D_{B_{2R}(x_0)}(\mathscr{E})$ is $\mathscr{L}$-superharmonic in $B_{2R}(x_0)$ and satisfies $u(x) \geqslant \delta > 0$ for almost any $x \in B_R(x_0)$. Then for any $p_0 > 0$*

$$\operatorname*{ess\text{-}inf}_{x \in B_{\mu R}(x_0)} u(x) \geqslant c\left(\fint_{B_R(x_0)} u(x)^{-p_0}\,\mathrm{d}x\right)^{-1/p_0}, \tag{15}$$

where $c > 0$ is independent of u, x_0, R, and δ.

Finally we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. For $s > 0$ let us write B_s instead of $B_s(x_0)$. First, we assume $u(x) \geqslant \delta$ for almost any $x \in B_R$ and some $\delta > 0$. Since $d \geqslant 2$, there exists $\widetilde{R} \in [R/2, R)$ such that $|\{x \in B_{\widetilde{R}}\colon u(x) \geqslant 1\}| \geqslant \frac{1}{8}|B_R|$, and at the same time $|\{x \in B_R \setminus B_{\widetilde{R}}\colon u(x) \geqslant 1\}| \geqslant \frac{1}{8}|B_R|$. Obviously, $\widetilde{R}$ depends on u. Due to Corollary 2.5, for any $p_0 > 0$

$$\operatorname*{ess\text{-}inf}_{x\in B_{\widetilde{R}/2}} u(x) \geqslant c\left(⨍_{B_{\widetilde{R}}} u(x)^{-p_0}\,\mathrm{d}x\right)^{-1/p_0}. \tag{16}$$

Set $\theta = 4$ and $S = \widetilde{R}$. A simple observation assures that there is $\gamma \in (0,1)$ with $\int_{\mathbb{R}^d\setminus B_{\widetilde{R}}} u(x)k(x,y)\,\mathrm{d}x \geqslant 0$ for almost any $y \in B_{\widetilde{R}}$. Lemma 2.3 implies that there exist $\bar{p} \in (0,1)$ and $c > 0$ such that

$$\left(⨍_{B_{\widetilde{R}}} u(x)^{\bar{p}}\,\mathrm{d}x\right)^{1/\bar{p}}\mathrm{d}x \leqslant c\left(⨍_{B_{\widetilde{R}}} u(x)^{-\bar{p}}\,\mathrm{d}x\right)^{-1/\bar{p}}.$$

Together with estimate (16), this implies

$$\operatorname*{ess\text{-}inf}_{x\in B_{\widetilde{R}/2}} u(x) \geqslant c\left(\frac{1}{|B_{\widetilde{R}}|}\int_{B_{\widetilde{R}}} u(x)^{\bar{p}}\,\mathrm{d}x\right)^{1/\bar{p}} \geqslant c\left(\frac{1}{|B_{\widetilde{R}}|}\int_{B_{\widetilde{R}}\cap\{u\geqslant 1\}} u(x)^{\bar{p}}\,\mathrm{d}x\right)^{1/\bar{p}} \geqslant c\left(\frac{|B_{\widetilde{R}}\cap\{u\geqslant 1\}|}{|B_R|}\right)^{1/\bar{p}} \geqslant c.$$

As a trivial consequence, $\operatorname{ess\text{-}inf}_{x\in B_{R/4}} u(x) \geqslant c$ and the assertion follows. If there is no $\delta > 0$ with $u(x) \geqslant \delta$ for almost any $x \in B_R$, we choose $v(x) = u(x) + \delta$. Next, we apply the same proof to v. In the limit $\delta \to 0$, we obtain the desired result. The proof of Theorem 2.1 is complete. □

Theorem 2.1 implies our main result, Theorem 2.6. This can be seen using the standard technique of oscillation reduction, see [9]. Concerning other approaches to Hölder regularity for related non-local problems see [5,1]. Here is our main result:

Theorem 2.6. *There exist $\beta \in (0,1)$, $c > 0$ such that for any $x_0 \in \mathbb{R}^d$, $R \in (0,1)$ and $u \in D_{B_{2R}(x_0)}(\mathscr{E})$ which is $\mathscr{L}$-harmonic in $B_{2R}(x_0)$ and almost any $x, y \in B_R(x_0)$*

$$\left|u(x) - u(y)\right| \leqslant c\|u\|_\infty\left(\frac{|x-y|}{R}\right)^{\beta}. \tag{17}$$

The constants β and c depend only on d, ν, M, η and α.

3. Extensions

There are several possible extensions of the results presented above. 1) Assumption (4) can be replaced by the significantly weaker assumption

$$\nu\mathbb{1}_{\{x-y\in M\}} \leqslant k(x,y)|x-y|^{d+\alpha} \leqslant \nu^{-1} \quad \text{for almost all } x, y \in \mathbb{R}^d \text{ with } |x-y| \leqslant 1,$$

where $M \subset \mathbb{R}^d$ is any set containing 0 and some cone with apex 0. 2) It is obvious that terms of lower order and so-called right-hand sides can be considered, too. 3) Currently, the author is working on a parabolic analog to Theorem 2.6. The methods of this work are quite general and do apply to the parabolic setting.

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