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Partial Differential Equations

Reaction–diffusion equations in space–time periodic media

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Abstract

This Note deals with reaction–diffusion in space–time periodic media. We state some conditions for the existence, uniqueness and large-time behavior of the solutions of such equations. These conditions are related to the two generalized principal eigenvalues associated with a linearized equation and we state some properties of these quantities. ***To cite this article: G. Nadin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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Résumé

Équations de réaction–diffusion en milieu périodique en temps et en espace. Cette Note traite des équations de réaction–diffusion en milieu périodique à la fois en temps et en espace. Nous établissons des conditions d'existence, d'unicité et de convergence en temps grand pour les solutions de telles équations. Ces conditions sont établies en fonctions de deux valeurs propres principales généralisées associées à une équation linéarisée. Nous établissons plusieurs propriétés de ces deux quantités. ***Pour citer cet article : G. Nadin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
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Version française abrégée

Cette Note traite de l'équation :

$$\partial_t u - \nabla\cdot\big(A(t,x)\nabla u\big) + q(t,x)\cdot\nabla u = f(t,x,u) \tag{1}$$

où la diffusion A, l'advection q et le terme de réaction f sont périodiques en temps et en espace. Cette équation intervient dans des modèles de dynamique des populations, de combustion et de génétique. La fonction f représente un taux d'accroissement local de la population et satisfait les hypothèses :

$$\begin{cases} \forall x\in\mathbb{R}^N, \forall t\in\mathbb{R},\ s\to f(t,x,s)/s \text{ décroit strictement sur } \mathbb{R}^{+*},\\ \exists M>0 \mid \forall x\in\mathbb{R}^N, \forall t\in\mathbb{R}, \forall s\geqslant M,\ f(t,x,s)\leqslant 0. \end{cases} \tag{2}$$

On suppose que $f(t,x,0)=0$ pour tout (t,x) et on note $\mu(t,x)=f'_s(t,x,0)$. On définit l'opérateur linéarisé au voisinage de 0 par :

$$\mathcal{L}\phi = \partial_t\phi - \nabla\cdot\big(A(t,x)\nabla\phi\big) - q(t,x)\cdot\nabla\phi - \mu(t,x)\phi.$$

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On peut associer à cet opérateur deux *valeurs propres principales généralisées* :

$$\lambda_1 = \sup\{\lambda \mid \exists\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N),\ \phi \text{ est périodique en } t,\ \phi > 0 \text{ et } \mathcal{L}\phi \geqslant \lambda\phi \text{ dans } \mathbb{R}\times\mathbb{R}^N\},$$
$$\lambda_1' = \inf\{\lambda \mid \exists\phi \in \mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N),\ \phi \text{ est périodique en } t,$$
$$\phi > 0 \text{ et } \mathcal{L}\phi \leqslant \lambda\phi \text{ dans } \mathbb{R}\times\mathbb{R}^N\}. \quad (3)$$

Les valeurs propres λ_1 et λ_1' conditionnent respectivement l'existence et l'unicité des solutions de (1). On montre que $\lambda_1' \leqslant \lambda_1$ est toujours vérifié, mais que l'égalité n'est pas vraie en général.

Théorème 0.1. *Si $\lambda_1' < 0$, l'équation* (1) *admet une solution strictement positive périodique $p \in C^{1,2}(\mathbb{R}\times\mathbb{R}^N)$. Si $\lambda_1' \geqslant 0$, alors la seule solution strictement positive entière bornée de* (1) *est* 0.

Théorème 0.2. *Si u et p sont deux solutions entières bornées de* (1) *telles que* : $\inf_{\mathbb{R}\times\mathbb{R}^N} u > 0$ *et* $\inf_{\mathbb{R}^N\times\mathbb{R}} p > 0$, *alors $u \equiv p$.*

Corollaire 0.3. *Si $\lambda_1' < 0$, alors il existe une unique solution périodique strictement positive de* (1).

Il est aisé de montrer que l'unicité dans la classe des solutions entières strictement positives bornées n'est pas vraie sans l'hypothèse d'existence d'une borne inférieure uniforme (voir [7]). Le théorème suivant établi un résultat d'unicité pour une classe de fonctions plus générale.

Théorème 0.4. *Soit u est une solution entière strictement positive bornée de* (1) *satisfaisant* :

$$\exists x_0 \in \mathbb{R}^N \mid \inf_{t\in\mathbb{R}} u(t, x_0) > 0.$$

Alors si $\lambda_1 < 0$, on a $u \equiv p$, où p est l'unique solution strictement positive et périodique.

Ce théorème permet de déterminer le comportement en temps grand des solutions du problème de Cauchy associé à (1) et à une donnée initiale u_0.

Théorème 0.5. *Soit $u_0 \in C^0(\mathbb{R}^N)$ une fonction positive, non-nulle et bornée. Alors* :

(1) *Si $\lambda_1 < 0$, on a $u(s,x) - p(s,x) \to 0$ et $\partial_t u(s,x) - \partial_t p(s,x) \to 0$ quand $s \to +\infty$ dans $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$.*
(2) *Si $\lambda_1' \geqslant 0$, on a $u(s,x) \to 0$ et $\partial_t u(s,x) \to 0$ quand $s \to +\infty$ dans $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

Il est possible que $\lambda_1' < 0 \leqslant \lambda_1$. Dans ce cas le comportement en temps grand dépend de la donnée initiale, comme le montre la proposition suivante :

Proposition 0.6. *Suppposons que $\lambda_1' < 0 \leqslant \lambda_1$.*

(1) *Si u_0 est une fonction continue telle que* $\inf_{\mathbb{R}^N} u_0 > 0$, *alors $u(s,x) - p(s,x) \to 0$ quand $s \to +\infty$.*
(2) *Si u_0 est une fonction continue positive à support compact, alors $u(s,x) \to 0$ quand $s \to +\infty$.*

Tous ces résultats sont prouvés dans [7,8].

1. Introduction

This Note is concerned with the equation:

$$\partial_t u - \nabla\cdot\big(A(t,x)\nabla u\big) + q(t,x)\cdot\nabla u = f(t,x,u) \quad (4)$$

with a periodic dependence in t and x. More precisely, we assume that there exists $T, L_1, \ldots, L_N > 0$ such that:

$$A(t+T,x)=A(t,x),\quad q(t+T,x)=q(t,x),\quad f(t+T,x,s)=f(t,x,s),$$
$$A(t,x+L_i)=A(t,x),\quad q(t,x+L_i)=q(t,x),\quad f(t,x+L_i,s)=f(t,x,s).\tag{5}$$

The function $f:\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^+\to\mathbb{R}$ is supposed to be of class $C^{\frac{\alpha}{2},\alpha}$ in (t,x) locally in u for a given $0<\alpha<1$, locally Lipschitz-continuous in u and of class C^1 on $\mathbb{R}\times\mathbb{R}^N\times[0,\beta]$ for a given $\beta>0$. The matrix field $A:\mathbb{R}\times\mathbb{R}^N\to S_N(\mathbb{R})$ is supposed to be of class $C^{\frac{\alpha}{2},1+\alpha}$. We suppose furthermore that A is uniformly elliptic and continuous: it exists some positive constants γ and Γ such that $\gamma I_N\leqslant A(t,x)\leqslant\Gamma I_N$ for all (t,x) for the positive matrix partial ordering. The drift term $q:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is supposed to be of class $C^{\frac{\alpha}{2},\alpha}$.

We assume that $\forall x,\forall t,\ f(t,x,0)=0$ and we set $\mu(t,x)=f'_s(t,x,0)$. We define the linearized operator in the neighborhood of 0:

$$\mathcal{L}\phi=\partial_t\phi-\nabla\cdot\big(A(t,x)\nabla\phi\big)-q(t,x)\cdot\nabla\phi-\mu(t,x)\phi.$$

This equation arises in population genetics, combustion and population dynamics models (see [11]). The heterogeneity is of particular interest in population dynamics models since the reaction term $f(t,x,u)$ represents an intrinsic growth rate which can depend on the environment. A typical nonlinearity is $f(t,x,u)=u(\mu(t,x)-\nu(t,x)u)$, where $\nu>0$ is a saturation term. In the results below, we will refer to the two following additional hypotheses on f:

$$\forall x\in\mathbb{R}^N,\forall t\in\mathbb{R},\quad s\to f(t,x,s)/s \text{ decreases on } \mathbb{R}^{+*},\tag{6}$$
$$\exists M>0\mid\forall x\in\mathbb{R}^N,\forall t\in\mathbb{R},\forall s\geqslant M,\quad f(t,x,s)\leqslant 0.\tag{7}$$

These hypotheses both reflect saturation effects, which are due to intraspecific competition for limited resources.

This model has been studied in [4–6], in the case of a bounded domain in x, eventually with periodicity boundary conditions in t. In all these papers, the authors proved that the existence of a positive periodic state, its uniqueness, its stability and the large-time behavior of the solutions of Cauchy problems were all determined by the sign of the principal eigenvalue associated with the linearized equation in the neighborhood of the homogeneous solution 0. In other terms, the instability of the null state yields the existence of a positive time periodic state and its global attractivity.

Recently, the case of a space-periodic and general unbounded domain has been investigated in [1,3]. This case is of particular interest since it allows us to search for *travelling fronts* that link the positive periodic state to 0. The existence of travelling waves in the case of a space–time periodic advection was proved in [10] and we wish to extend this result to space–time periodic growth rate and diffusion matrix, as it was done in [2] in the case of a time-homogeneous environment (see [9]). The first step is the study of the positive space–time periodic solutions of (4).

2. The associated eigenvalue problem

We expect to get existence and uniqueness conditions for the positive periodic solutions of Eq. (4) related to the sign of the principal eigenvalue associated with the operator:

$$\begin{aligned}\lambda_1&=\sup\big\{\lambda\mid\exists\phi\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N),\phi\text{ is periodic in }t,\phi>0\text{ and }\mathcal{L}\phi\geqslant\lambda\phi\text{ in }\mathbb{R}\times\mathbb{R}^N\big\},\\ \lambda_1'&=\inf\big\{\lambda\mid\exists\phi\in\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}\times\mathbb{R}^N),\phi\text{ is periodic in }t,\ \phi>0\text{ and }\mathcal{L}\phi\leqslant\lambda\phi\text{ in }\mathbb{R}\times\mathbb{R}^N\big\}.\end{aligned}\tag{8}$$

We need to state a few results related to these eigenvalue before investigating Eq. (4).

First of all, for any bounded smooth domain, one can define (see [6]) the principal eigenvalue $\lambda_1(\Omega)$ associated with the same operator $\mathcal{L}$ with periodic boundary conditions in t and Dirichlet boundary conditions in x. Then one can prove the following approximation result:

Proposition 2.1. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of nonempty open sets such that $\Omega_n\subset\Omega_{n+1}$, $\bigcup_{n\in\mathbb{N}}\Omega_n=\mathbb{R}^N$. Then $\lambda_1(\Omega_n)\searrow\lambda_1$ as $n\to+\infty$.*

Next, we are able to characterize the two generalized principal eigenvalues with the help of a family of periodic principal eigenvalues. Namely, for all $\alpha\in\mathbb{R}^N$, define $L_\alpha\phi=\mathrm{e}^{-\alpha\cdot x}\mathcal{L}(\mathrm{e}^{\alpha\cdot x}\phi)$, then one can define the periodic principal eigenelements associated to this operator by:

Proposition 2.2. *There exists a unique constant k_α such that there exists a positive space–time periodic function $\phi \in \mathbb{R} \times \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ that satisfies $L_\alpha \phi = k_\alpha \phi$. Moreover, such a function ϕ is unique up to multiplication by a positive constant.*

Theorem 2.3. *One has $\lambda_1 = \max_{\alpha \in \mathbb{R}^N} k_\alpha$ and $\lambda_1' = k_0$.*

These two generalized eigenvalues are not equal in general. The equality holds in the case of a time-homogeneous environment without drift. We are able to extend this result if the coefficients exhibit a common symmetry axis in t or x:

Proposition 2.4. *If A and μ have a common symmetry axis in t or in x, in other words if*:

$$\begin{aligned} &\exists x_0 \mid \forall t, x,\ A(t, x_0 + x) = A(t, x_0 - x) \text{ and } \mu(t, x_0 + x) = \mu(t, x_0 - x) \text{ and } q(t, x_0 + x) = q(t, x_0 - x), \\ &\text{or if } \exists t_0 \mid \forall t, x,\ A(t_0 + t, x) = A(t_0 - t, x) \text{ and } \mu(t_0 + t, x) = \mu(t_0 - t, x) \text{ and } q(t_0 + t, x) = q(t_0 - t, x), \end{aligned} \tag{9}$$

and if q can be written $q = A\nabla Q$ where $Q \in \mathcal{C}^{0,1}(\mathbb{R} \times \mathbb{R}^N)$ with $\int_{(0,T)\times C} A^{-1} q = 0$, then $\lambda_1' = \lambda_1$.

All the results above are proved in [8].

3. Existence and uniqueness results

Theorem 3.1. *If $\lambda_1' < 0$ and if hypothesis* (7) *is satisfied, then there exists a positive periodic solution $p \in C^{1,2}(\mathbb{R} \times \mathbb{R}^N)$ of Eq.* (4). *If $\lambda_1' \geqslant 0$ and if hypothesis* (6) *is satisfied, then the only nonnegative bounded and entire solution of* (4) *is* 0.

This means that the existence of a positive periodic solution only depends on the stability of the solution 0. We next state the following uniqueness result for some entire solutions, that is to say solutions that are defined on the whole space $\mathbb{R} \times \mathbb{R}^N$, and its corollary:

Theorem 3.2. *We assume that* (7) *is satisfied. Then if u and p are two bounded nonnegative entire solutions of* (4) *such that $\inf_{\mathbb{R}\times\mathbb{R}^N} u > 0$ and $\inf_{\mathbb{R}^N\times\mathbb{R}} p > 0$, one has $u \equiv p$.*

Corollary 3.3. *If $\lambda_1' < 0$ and* (7) *is satisfied, there is a unique positive periodic continuous solution of Eq.* (4).

This means that the existence and uniqueness problem for the space–time positive periodic solutions of (4) is fully determined by the sign of λ_1'. The condition $\inf_{\mathbb{R}\times\mathbb{R}^N} u > 0$ is required in order to get uniqueness, otherwise, using travelling waves solutions, one can build an infinity of entire solutions. Thus, the next step is the identification of the entire solutions that satisfy this condition.

Theorem 3.4. *If u is a positive bounded entire solution of* (4) *that satisfies*:

$$\exists x_0 \in \mathbb{R}^N \mid \inf_{t\in\mathbb{R}} u(t, x_0) > 0 \tag{10}$$

then if $\lambda_1 < 0$ and (7) *is satisfied, one has $\inf_{t\in\mathbb{R}, x\in\mathbb{R}^N} u(t, x) > 0$ and $u \equiv p$.*

The proof relies on Proposition 2.1. Namely, using (6), we compare the entire solution with the solution of the time-periodic linearized problem in a ball with Dirichlet boundary conditions. All the results above are proved in [7].

4. Large time behavior

We are now interested in the large-time behavior of the solution u of the Cauchy problem:

$$\begin{cases} \partial_t u - \nabla \cdot (A(t, x)\nabla u) + q(t, x) \cdot \nabla u = f(x, t, u), \\ u(0, x) = u_0(x). \end{cases} \tag{11}$$

The uniqueness result for entire solutions enables us to prove the following asymptotic convergence when $t \to +\infty$:

Theorem 4.1. *Let $u_0 \in C^0(\mathbb{R}^N)$, nonnegative, bounded and $u_0 \neq 0$. If hypotheses* (6) *and* (7) *are satisfied, then*:

(1) *If $\lambda_1 < 0$, then $u(s,x) - p(s,x) \to 0$ and $\partial_t u(s,x) - \partial_t p(s,x) \to 0$ as $s \to +\infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, where p is the unique positive solution of* (4).
(2) *If $\lambda'_1 \geqslant 0$, then $u(s,x) \to 0$ and $\partial_t u(s,x) \to 0$ as $s \to +\infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$.*

If $\lambda'_1 < 0 \leqslant \lambda_1$, then the asymptotic behavior of the solutions of the Cauchy problem is not known in general. The following propositions yield that there is no possible general conclusion to this issue and the asymptotic behavior depends on the initial data in this case.

Proposition 4.2. *Let u_0 be a continuous nonnegative function, $u_0 \neq 0$. Assume that* (6) *and* (7) *are satisfied and $\lambda_1 > 0$. If there exists $\alpha \in \mathbb{R}^N$ such that $k_\alpha > 0$ and*:

$$\exists C > 0 \mid \forall x \in \mathbb{R}^N, \quad \left|u_0(x)\right| \leqslant C e^{\alpha \cdot x},$$

then the solution of the associated Cauchy problem (11) *has the following asymptotic behavior*:

$$u(s,x) \to 0 \quad \text{as } s \to +\infty \text{ uniformly on the compact subsets of } \mathbb{R} \times \mathbb{R}^N.$$

Remark 1. An initial data u_0 with compact support satisfies the required condition.

Proposition 4.3. *If $\lambda'_1 < 0$ and u_0 is a continuous bounded function such that*:

$$\exists B \in \mathbb{R}, \kappa > 0 \mid \forall x < B, u_0(x) \geqslant \kappa\, \mathrm{e}^{\alpha \cdot x}$$

where $\alpha \in \mathbb{R}^N$ is such that $k_\alpha < 0$ and $t \mapsto k_{t\alpha}$ is nondecreasing in the neighborhood of 1. *Then*:

$$u(s,x) - p(s,x) \to 0 \quad \text{as } s \to +\infty \text{ in } \mathcal{C}^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^N).$$

All the results above are proved in [7].

References

[1] H. Berestycki, F. Hamel, L. Roques, Analysis of the periodically fragmented environment model: I – Influence of periodic heterogeneous environment on species persistence, J. Math. Biol. 51 (2005) 75–113.
[2] H. Berestycki, F. Hamel, L. Roques, Analysis of the periodically fragmented environment model: II – Biological invasions and pulsating travelling fronts, J. Math. Pures Appl. 85 (2005) 1101–1146.
[3] H. Berestycki, F. Hamel, L. Rossi, Liouville-type result for semilinear elliptic equations in unbounded domains, Ann. Mat. Pura Appl. (2006).
[4] R.S. Cantrell, C. Cosner, Diffusive logistic equations with indefinite weights: population models in disrupted environments I, Proc. Roy. Soc. Edinburgh 112 (1989) 293–318.
[5] R.S. Cantrell, C. Cosner, Diffusive logistic equations with indefinite weights: population models in disrupted environments II, SIAM J. Math. Anal. 22 (4) (1991) 1043–1064.
[6] P. Hess, Periodic-Parabolic Boundary Value Problems and Positivity, Pitman Research Notes in Mathematics Series, vol. 247, Longman Scientific and Technical, 1991.
[7] G. Nadin, Existence and uniqueness of the solution of a space–time periodic reaction–diffusion equation, 2007, submitted for publication.
[8] G. Nadin, The principal eigenvalue of a space–time periodic parabolic operator, 2007, submitted for publication.
[9] G. Nadin, Travelling fronts in space–time periodic media, 2007, in preparation.
[10] J. Nolen, M. Rudd, J. Xin, Existence of kpp fronts in spatially-temporally periodic advection and variational principle for propagation speeds, Dynam. Partial Differential Equations 2 (1) (2005) 1–24.
[11] N. Shigesada, K. Kawasaki, Biological Invasions: Theory and Practice, Oxford Series in Ecology and Evolution, Oxford University Press, Oxford, 1997.