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Littlewood–Paley and Lusin functions on stratified groups

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Abstract

In this Note, we define the Littlewood–Paley and Lusin functions associated with the sub-Laplacian operator on stratified groups. The L^p $(1 < p < \infty)$ boundedness of Littlewood–Paley and Lusin functions are proved. ***To cite this article: J. Zhao, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Les fonctions de Littlewood–Paley et Lusin sur les groupes stratifiés. Dans cette Note, nous définissons les fonctions de Littlewood–Paley et de Lusin sur les groupes stratifiés. Nous prouvons que pour $1 < p < \infty$, elles sont bornées sur L^p. ***Pour citer cet article : J. Zhao, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Preliminaries

In classical harmonic analysis, the Littlewood–Paley functions play an important role in the study of non-tangential convergence of Fatou type and the boundedness of Riesz transforms and multipliers [9–11].

In [9], Stein extended the L^p boundedness of the vertical Littlewood–Paley $\mathcal{G}$-function to the context of compact Lie groups, and the L^p boundedness of the horizontal Littlewood–Paley g-function to a general setting of symmetric Markov semigroups, for $1 < p < \infty$. For the latter see [8] and the references therein. These facts have been generalised further. One direction is the Littlewood–Paley theory on Coifman–Weiss's spaces of homogeneous type, see [5]. Another direction is the study of the Littlewood–Paley functions on non-compact complete Riemannian manifolds, in connection with the study Riesz transforms: some results have been obtained by N. Lohoué for Cartan–Hadamard manifolds and non-amenable Lie group, see [6,7], and by J.C. Chen for Riemannian manifolds with non-negative Ricci curvature, see [1]. T. Coulhon, X. Duong, X.D. Li studied Littlewood–Paley-Stein functions on complete Riemannian manifolds for $1 \leqslant p \leqslant 2$, see [2]. The literature is so vast that we do not give exhaustive references.

The aim of this Note is to study the L^p boundedness of Littlewood–Paley functions defined on the stratified group, where $1 < p < \infty$. To prove this, we also need to define the Littlewood–Paley g_λ^* function and prove its L^p boundedness. The difficult point is to prove the L^p boundedness where $2 < p < \infty$.

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First let us to recall some properties on the stratified group which we will use in the sequel, for more details, see [3,4].

A stratified group is a simply connected nilpotent Lie group G endowed with a graded Lie algebra $\mathfrak{g}$, which is decomposed into a direct sum of subspaces $V_j : \mathfrak{g} = V_1 \oplus \cdots \oplus V_m$ such that $V_{j+1} = [V_1, V_j]$ for every $j < m$, and $[V_1, V_m] = \{0\}$.

The elements of $\mathfrak{g}$ will be considered as left-invariant vector fields on G, and we fix a basis $X_1, \ldots, X_n$ for $V_1 \subset \mathfrak{g}$. The operator $\Delta = -\sum_{j=1}^n X_j^2$ is called the sub-Laplacian of G and the associated gradient is defined by $\nabla = (X_1, \ldots, X_n)$.

We define a one-parameter family $\{\gamma_r\colon\ r > 0\}$ of automorphisms of $\mathfrak{g}$, called dilation, by the formula

$$\gamma_r\left(\sum_1^m Y_j\right) = \sum_1^m r^j Y_j \quad (Y_j \in V_j).$$

The dilations $\{\gamma_r\}$ on $\mathfrak{g}$ induce automorphisms of G, still called dilations and defined by $\phi_r(x) = rx = \exp(\gamma_r(\exp^{-1} x)), r > 0, x \in G$.

The number $Q = \sum_1^m j(\dim V_j)$ is called the homogeneous dimension of G, since $\mathrm{d}(rx) = r^Q \mathrm{d}x$ for $r > 0$, where $\mathrm{d}x$ is bi-invariant Haar measure on G.

Let $Y \to \|Y\|$ be a Euclidean norm on $\mathfrak{g}$. If $x \in G$, we set $\|x\| = \|\exp^{-1} x\|$. Let $x \to |x|$ on G be a homogeneous norm defined by

$$\left|\exp \sum_1^m Y_j\right| = \left(\sum_1^m \|Y_j\|^{\frac{2m!}{j}}\right)^{\frac{1}{2m!}} \qquad (Y_j \in V_j).$$

The homogeneous norm is continuous on G, C^∞ on $G - \{0\}$, homogeneous of degree 1, and satisfies (a) $|x| > 0$ if $x \neq 0$, (b) $|x| = |x^{-1}|$, where m is the number of steps in the stratification of $\mathfrak{g}$.

Consider the group $G \times \mathbb{R}$, whose Lie algebra has a natural stratification $\bigoplus_1^m W_j$, where W_1 is the span of V_1 and ∂_t and $W_j = V_j$ for $j > 1$.

The corresponding dilations are given by $r(x,t) = (rx, rt)$, the second factor being the ordinary multiplication, and the homogeneous dimension of $G \times \mathbb{R}$ is $Q + 1$.

The sub-Laplacian of $G \times \mathbb{R}$ is defined by $\Delta_H = -\frac{\partial^2}{\partial t^2} - \sum_{j=1}^n X_j^2$, and the associated gradient is defined by $\nabla_H = (\frac{\partial}{\partial t}, X_1, \ldots, X_n)$.

Before defining the Poisson kernel, we give the following facts due to [3,4].

There is a unique C^∞ function K on $G \times \mathbb{R} - \{(0,0)\}$ which satisfies (a) $K(rx, rt) = r^{1-Q} K(x,t)$, (b) $\Delta_H K$ is the Dirac distribution at $(0,0) (Q > 1)$. (This result holds only if $Q > 1$. If $Q = 1$, then $G = \mathbb{R}$ and Δ_H is minus the classical Laplacian on $\mathbb{R}^2$, and we take K to be the usual logarithmic potential.) K is real and satisfies $K(x,t) = K(x^{-1}, -t)$, $K(x,t) = K(x,-t)$, $K(x,t) = K(x^{-1}, t)$.

Now we define the Poisson kernel $p(x,t) = p_t(x)$ by $p(x,t) = A^{-1} q(x,t), (t > 0, x \in G)$, where $q(x,t) = \partial_t K(x,t)$, and $A = \int_G q(x,t)\,\mathrm{d}x = \int_G q(x,1)\,\mathrm{d}x \neq 0$, then the corresponding operator P_t is defined by $P_t f(x) = p_t * f(x)$.

We have the following estimate of the Poisson kernel. For further properties of the Poisson kernel, see [4].

Proposition 1.1. $p_t(x) = p(x,t) \leqslant \frac{Ct}{(t+|x|)^{Q+1}}, t > 0, x \in G$.

Now we define the Littlewood–Paley and Lusin functions as follows:

$$g(f)(x) = \left(\int_0^{+\infty} \left|\nabla_H (P_t f)(x)\right|^2 t\,\mathrm{d}t\right)^{\frac{1}{2}},$$

$$g_\lambda^*(f)(x) = \left(\int_0^{+\infty}\int_G \left(\frac{t}{t+|y|}\right)^{\lambda Q} \left|\nabla_H (P_t f)(xy)\right|^2 t^{-Q+1}\,\mathrm{d}y\,\mathrm{d}t\right)^{\frac{1}{2}},$$

$$S(f)(x) = \left(\int_0^{+\infty} \int_{|x^{-1}y| \leqslant t} \left|\nabla_H (P_t f)(y)\right|^2 t^{-Q+1} \, \mathrm{d}y \, \mathrm{d}t \right)^{\frac{1}{2}}.$$

We will study the boundedness of Littlewood–Paley g-function in the following section.

2. The Littlewood–Paley g-function

The basic results for g are the following, and the proof of the second theorem is more complicated.

Theorem 2.1. *g is $L^p(G)$ bounded, where $1 < p \leqslant 2$.*

Theorem 2.2. *g is $L^p(G)$ bounded, where $2 < p < \infty$.*

Following Stein's argument, first we prove four lemmas.

Lemma 2.3.

$$\Delta_H\left(u^p\right) = p(p-1)u^{p-2}|\nabla_H u|^2,$$

where $u(x,t) = (P_t f)(x)$, $x \in G$, $t > 0$, $1 < p \leqslant 2$.

Using the fact that $\Delta_H u = 0$, we can prove it easily.

Lemma 2.4.

$$\int_0^{+\infty} \int_G t \Delta_H (P_t f)(x) \, \mathrm{d}x \, \mathrm{d}t = \int_G f(x) \, \mathrm{d}x.$$

Lemma 2.5. $\sup_{t>0} |(P_t f)(x)| \leqslant CAM(f)(x)$, *where $f \in L^p(G)$, $p \geqslant 1$, $A = \int_G \psi(x) \, \mathrm{d}x$, $\psi(x) = \sup_{|y| \geqslant |x|} |Q(y)|$, $Q(x) = \frac{1}{(1+|x|)^{Q+1}}$.*

Lemma 2.6. *Let $f \in L^p(G)$, $p \geqslant \mu$, $\mu \geqslant 1$, then*

$$\left|(P_t f)(xy)\right| \leqslant C\left(1 + \frac{|y|}{t}\right)^Q M(f)(x),$$

more generally, we have

$$\left|(P_t f)(xy)\right| \leqslant C_\mu\left(1 + \frac{|y|}{t}\right)^{\frac{Q}{\mu}} M_\mu(f)(x),$$

where $M_\mu(f)(x) = (\sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^\mu \, \mathrm{d}y)^{\frac{1}{\mu}}$.

By Lemmas 2.4 and 2.5, we can prove Theorem 2.1. To prove Theorem 2.2, we need to prove the following vector-valued singular integral theorem first.

Theorem 2.7. *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two Hilbert spaces, suppose that*

$$Tf(x) = \int_G K\left(x^{-1}y\right) f(y) \, \mathrm{d}y,$$

is a bounded operator from $L^2(G, \mathcal{B}_1)$ into $L^2(G, \mathcal{B}_2)$. Assume that K satisfies

$$\left|\nabla K(x)\right|_{\mathcal{B}_1 \to \mathcal{B}_2} \leqslant \frac{C}{|x|^{Q+1}},$$

then there exists a constant A_p such that $\|Tf\|_p \leqslant A_p \|f\|_p$, $1 < p < +\infty$.

3. Lusin function

In this section, we will prove the L^p-boundedness of the Lusin function. By the definition, it is easy to see that $S(f) \leqslant C g_\lambda^*(f)$. The main theorem of this part is the following:

Theorem 3.1. $S(f) \in L^p(G)$, $f \in L^p(G)$, $1 < p < \infty$.

The proof of this theorem depends on the following theorem.

Theorem 3.2. *$g_\lambda^*(f) \in L^p(G)$, where $f \in L^p(G)$, $1 < p < \infty$.*

By Lemmas 2.3, 2.4, 2.6, we can prove this theorem.

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