



Partial Differential Equations

# On perfect fluids with bounded vorticity

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## Abstract

This Note is devoted to studying the incompressible Euler equations. First, we prove global existence for three-dimensional axisymmetric solutions *without swirl* under a regularity assumption which is very close to the one which has been introduced in the two-dimensional setting by V. Yudovich (1963). Second, we state uniqueness in the general  $N$ -dimensional case for bounded solutions with bounded vorticity. **To cite this article:** R. Danchin, *C. R. Acad. Sci. Paris, Ser. I 345 (2007)*.

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## Résumé

**Une remarque sur les fluides parfaits à tourbillon borné.** On s'intéresse aux équations d'Euler incompressibles. On établit d'abord l'existence globale pour des données axisymétriques sans swirl en dimension trois, vérifiant des hypothèses très proches de celles de V. Yudovich (1963) en dimension deux. On démontre ensuite un résultat général d'unicité en dimension  $N$  dans la classe des solutions bornées à tourbillon borné. **Pour citer cet article :** R. Danchin, *C. R. Acad. Sci. Paris, Ser. I 345 (2007)*.

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## 1. Introduction

We are concerned with the Cauchy problem for the  $N$ -dimensional incompressible Euler equations:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0, \\ \operatorname{div} v = 0 \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (E)$$

In dimension two, global existence and uniqueness has been stated by V. Yudovich in [8] for data with bounded vorticity. The proof relies on the following facts: first, the vorticity  $\omega$  associated to  $v$  (which, in dimension two, reduces to a scalar function) is transported by the flow hence has constant  $L^\infty$  norm; second, having  $\omega$  bounded and  $\operatorname{div} v = 0$  implies that  $v$  is quasi-Lipschitz. One can then prove a stability estimate in energy norm by taking advantage of a generalized Gronwall lemma.

In dimension three, the problem of global solvability is much more involved. Indeed the vorticity (which may be identified with a solenoidal vector-field) is transported by the flow as a vector-field, namely

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v.$$

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Hence the  $L^\infty$  norm of  $\omega$  may grow in time and the problem of global solvability has remained unsolved. We focus on the case of axisymmetric data *without swirl*: the initial velocity  $v^0$  is assumed to be given in cylindrical coordinates  $(r, \theta, z)$  by  $v^0(r, \theta, z) = v_r^0(r, z)e_r + v_z^0(r, z)e_z$  where  $e_r$  (resp.  $e_z$ ) stands for the unit outer radial (resp. vertical) vector. The vorticity  $\omega^0$  then reduces to

$$\omega^0(r, \theta, z) = \omega_\theta^0(r, z)e_\theta \quad \text{with } \omega_\theta := \partial_z v_r - \partial_r v_z \text{ and } e_\theta = e_z \times e_r.$$

Global well-posedness for (E) with axisymmetric data has been first proved by M. Ukhovskii and V. Yudovich in [6] under the additional assumption that  $v^0 \in H^1$  and that the initial vorticity  $\omega^0$  is in  $L^\infty$  and satisfies  $r^{-1}\omega^0 \in L^2 \cap L^\infty$  (see also [4]). The proof relies on the fact that the quantity  $r^{-1}\omega_\theta$  is transported by the flow, hence plays the same role as the vorticity in dimension two. In terms of regularity in Sobolev spaces, [6]’s assumptions are stronger than those which are needed to have local well-posedness in dimension three. Indeed, local existence holds true in  $H^s$  whenever  $s > 5/2$  whereas  $s > 7/2$  is required for having  $r^{-1}\omega^0$  in  $L^\infty$  for all  $v^0 \in H^s(\mathbb{R}^3)$ . This gap has been filled in by T. Shirota and T. Yanagisawa in [5].

In the present Note, we aim at getting a global existence result as close as possible to that of Yudovich in dimension two. As a matter of fact, we strive for a global result in a functional space (for the vorticity) which has the same *scaling invariance* as  $L^\infty(\mathbb{R}^3)$ . This is achieved in the following statement:

**Theorem 1.1.** *Let  $\omega^0$  be an axisymmetric function in  $L^{3,1} \cap L^\infty$  such that  $r^{-1}\omega^0 \in L^{3,1}$ . Let  $v^0$  be a bounded axisymmetric solenoidal vector-field with vorticity  $\omega^0 e_\theta$ . Then Euler equations (E) have a global solution  $(v, \nabla p)$  with  $v \in C_w(\mathbb{R}; C_\star^1)$  and  $\nabla p \in \mathcal{C}(\mathbb{R}; C^{1-\varepsilon})$  for all  $\varepsilon > 0$ . Besides, the vorticity satisfies*

$$\omega \in L_{\text{loc}}^\infty(\mathbb{R}; L^{3,1} \cap L^\infty) \quad \text{and} \quad \|r^{-1}\omega(t)\|_{L^{3,1}} = \|r^{-1}\omega^0\|_{L^{3,1}} \quad \text{for all } t \in \mathbb{R}.$$

Above, the notation  $C_\star^1$  stands for the Zygmund space of bounded continuous functions  $f$  such that there exists a constant  $\Lambda$  so that  $|f(x+y) + f(x-y) - 2f(x)| \leq \Lambda|y|^2$  for all  $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ , and  $L^{3,1}$  is the Lorentz space which may be defined by real interpolation by  $L^{3,1} := (L^\infty, L^1)_{\frac{1}{3}, 1}$ . Note that the set of bounded functions  $f$  such that  $r^{-1}f$  belongs to  $L^{3,1}$  has the same scaling as  $L^\infty$ .

For stating our uniqueness result, we need to introduce the set  $L_L^\infty$  of locally bounded functions  $f$  such that  $\sup_{x \in \mathbb{R}^N} (\log(2 + |x|))^{-1} |f(x)| < \infty$ . Our statement reads:

**Theorem 1.2.** *Let  $(v^1, p^1)$  and  $(v^2, p^2)$  solve the  $N$ -dimensional incompressible Euler equations on  $[0, T]$ . Assume that  $v^1, v^2$  belong to  $L^\infty([0, T] \times \mathbb{R}^N) \cap L^1([0, T]; C_\star^1)$  and that  $p^1, p^2$  are in  $L^\infty([0, T]; L_L^\infty)$ .*

*If, in addition,  $v^1(0) = v^2(0)$  then  $v^1 \equiv v^2$  on  $[0, T] \times \mathbb{R}^N$ .*

Note that we do not need to make any decay assumption in Theorem 1.2. In particular, the assumptions are fulfilled if  $v, p$  and  $\omega$  are in  $L^\infty([0, T] \times \mathbb{R}^3)$ . Hence Theorem 1.2 is *not* a by-product of Vishik’s statement in [7] where some decay at infinity for the vorticity is needed.

## 2. The proof of global existence

The proof of Theorem 1.1 follows a standard scheme: solving (E) for regularized data, proving global a priori estimates for smooth axisymmetric solutions, then passing to the limit. Steps one and three are classical (it is only a matter of choosing a mollifier which preserves the axisymmetric structure). So we focus on the second step which relies on the following proposition:

**Proposition 2.1.** *There exists a constant  $C$  such that*

$$\forall t \in [0, T], \quad \|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} \exp(Ct \|r^{-1}\omega^0\|_{L^{3,1}}).$$

Proposition 2.1 is based on Lemmas 2.2 and 2.3 below.

**Lemma 2.2.** *There exists a constant C such that*

$$\left\| \frac{v_r}{r} \right\|_{L^\infty} \leq C \left\| \frac{\omega_\theta}{r} \right\|_{L^{3,1}}.$$

**Proof.** According to Lemma 1 in [5], there exists a constant C such that

$$|v_r(x)| \leq C \left( \int_{|x'-x|<r} \frac{|\omega_\theta(x')|}{|x-x'|^2} dx' + r \int_{|x'-x|\geq r} \frac{|\omega_\theta(x')|}{|x-x'|^3} dx' \right).$$

On one hand, if  $x' \in \mathbb{R}^3$  is such that  $|x' - x| \leq r$ , then  $r' \leq 2r$ . On the other hand, if  $|x' - x| > r$  then we have  $r'|x - x'|^{-1} \leq 2$ . Therefore,

$$\frac{|v_r(x)|}{r} \leq 2C \int \frac{1}{|x-x'|^2} \frac{|\omega_\theta(x')|}{r'} dx'.$$

Since  $L^{\frac{3}{2},\infty}$  is the dual set of  $L^{3,1}$  and  $y \mapsto |y|^{-2}$  belongs to  $L^{\frac{3}{2},\infty}$ , we get the desired result.  $\square$

**Lemma 2.3.** *Let a satisfy  $\partial_t a + \operatorname{div}(va) = f$  with v a smooth divergence free vector-field. Then we have*

$$\forall t \in [0, T], \quad \|a(t)\|_{L^{3,1}} \leq \|a(0)\|_{L^{3,1}} + \int_0^t \|f(\tau)\|_{L^{3,1}} d\tau. \tag{1}$$

Furthermore, equality holds if  $f \equiv 0$ .

**Proof.** One can assume with no loss of generality that  $f \equiv 0$ . Introducing the flow  $\psi$  of  $v$ , we get  $a(t, \psi(t, x)) = a(0, x)$  for all  $(t, x) \in [0, T] \times \mathbb{R}^3$ . Therefore

$$\{y \in \Omega \mid |a(t, y)| > \lambda\} = \psi_t(\{x \in \Omega \mid |a(0, x)| > \lambda\}).$$

Due to  $\operatorname{div} v = 0$ , both sets have the same measure. Now,  $L^{3,1}$  coincides with the set of functions  $g$  such that

$$\|g\|_{L^{3,1}} := \int_0^\infty \tau^{1/3} g^*(\tau) \frac{d\tau}{\tau} < \infty,$$

where

$$g^*(\tau) = \inf\{\lambda \in \mathbb{R}^+ \mid |\{x \in \Omega \mid |g(x)| > \lambda\}| \leq \tau\}.$$

This completes the proof.  $\square$

One can now prove Proposition 2.1. As  $(\partial_t + v \cdot \nabla)(r^{-1}\omega_\theta) = 0$ , Lemma 2.3 gives  $\|r^{-1}\omega(t)\|_{L^{3,1}} = \|r^{-1}\omega^0\|_{L^{3,1}}$ . As  $(\partial_t + v \cdot \nabla)\omega_\theta = r^{-1}v_r\omega_\theta$ , Lemma 2.2 and Gronwall inequality yield the result.

### 3. The proof of uniqueness

We shall use the so-called Bony decomposition (introduced in [1]) for the product of two distributions:

$$fg = T_f g + T_g f + R(f, g).$$

The paraproduct operator  $T$  is defined by  $T_f g := \sum_q S_{q-1} f \Delta_q g$  and the remainder operator  $R$ , by  $R(f, g) := \sum_q \Delta_q f (\Delta_{q-1} g + \Delta_q g + \Delta_{q+1} g)$ . See [2] for the definition of operators  $\Delta_q$  and  $S_q$ .

According to Corollary 2.5.1 in [2], that  $(v, p)$  be in  $L^\infty([0, T] \times \mathbb{R}^N) \times L^\infty([0, T]; L_L^\infty)$  guarantees that  $\nabla p = \Pi(v, v)$  where  $\Pi$  stands for the bilinear operator defined by

$$\Pi(v, w) := -\nabla(-\Delta)^{-1}(T_{\partial_i w^j} \partial_j v^i + T_{\partial_i v^i} \partial_i w^j) + \nabla T_{i,j} R(v^i, w^j). \tag{2}$$

The summation convention over repeated indices has been used, and  $T_{i,j}$  stands for the linear operator introduced by J.-Y. Chemin in [2], Theorem 2.5.1. We shall just use the fact that  $\nabla T_{i,j}$  maps  $C^{1-\varepsilon}$  in  $C^{-\varepsilon}$  for all  $\varepsilon \in (0, 1)$  (see [2], Chapter 2 for the definition of Hölder spaces with negative exponents).

We claim that the bilinear operator  $\Pi$  satisfies the following estimate for all  $\varepsilon \in (0, 1)$ :

$$\forall q \geq -1, \quad \|\Delta_q \Pi(v, w)\|_{L^\infty} \leq C(q+2)2^{q\varepsilon} \min(\|v\|_{C^{-\varepsilon}} \|w\|_{C^1_\star}, \|w\|_{C^{-\varepsilon}} \|v\|_{C^1_\star}). \tag{3}$$

This may be easily deduced from (2). Indeed: we have  $\Delta_q(T_{\partial_i w^j} \partial_j v^i) = \sum_{q'=q-4}^{q+4} \Delta_q(S_{q'-1} \partial_i w^j \Delta_{q'} \partial_j v^i)$  so that, because  $\nabla(-\Delta)^{-1}$  is an homogeneous operator of degree  $-1$ ,

$$\|\Delta_q \nabla(-\Delta)^{-1}(S_{q'-1} \partial_i w^j \Delta_{q'} \partial_j v^i)\|_{L^\infty} \leq C2^{-q} \|S_{q'-1} \nabla w\|_{L^\infty} \|\Delta_{q'} \nabla v\|_{L^\infty} \leq C(q+2)2^{q\varepsilon} \|w\|_{C^1_\star} \|v\|_{C^{-\varepsilon}}.$$

Next, standard continuity results for the paraproduct (see e.g. [2]) yield

$$\|\Delta_q \nabla(-\Delta)^{-1} T_{\partial_i w^j} \partial_j v^i\|_{L^\infty} \leq C2^{q\varepsilon} \|v\|_{C^1_\star} \|w\|_{C^{-\varepsilon}}.$$

Similar inequalities may be proved for the second term in (2). Finally, the remainder operator  $R$  maps  $C^{-\varepsilon} \times C^1_\star$  in  $C^{1-\varepsilon}$  provided  $\varepsilon < 1$ . Since  $\nabla T_{i,j}$  maps  $C^{1-\varepsilon}$  in  $C^{-\varepsilon}$ , one can now conclude to inequality (3). Finally, it may be easily shown that  $\Delta_q(v \cdot \nabla w)$  satisfies (3). It is only a matter of using that

$$v \cdot \nabla w^i = T_{v^j} \partial_j w^i + \partial_j(v^j, w^i) + T_{\partial_j w^i} v^j.$$

We are now ready to prove Theorem 1.2. First, we notice that  $\delta v := v^2 - v^1$  satisfies

$$\partial_t \delta v + v^2 \cdot \nabla \delta v = \Pi(\delta v, v^1) + \Pi(v^2, \delta v) - \delta v \cdot \nabla v^1. \tag{4}$$

With inequality (3) at our disposal, Eq. (4) may be seen as a transport equation associated to a vector-field with coefficients in  $L^1([0, T]; C^1_\star)$  and a right-hand side  $\delta f$  which satisfies

$$\|\Delta_q \delta f\|_{L^\infty} \leq C(q+2)2^{q\varepsilon} (\|v^1\|_{C^1_\star} + \|v^2\|_{C^1_\star}) \|\delta v\|_{C^{-\varepsilon}} \quad \text{for all } \varepsilon \in (0, 1). \tag{5}$$

We notice that the function  $t \mapsto V(t) := \|v^1(t)\|_{C^1_\star} + \|v^2(t)\|_{C^1_\star}$  belongs to  $L^1([0, T])$ . Therefore, by virtue of Lemma 2.5 in [3], there exists some constant  $C$  such that

$$\|\delta v(t)\|_{C^{-\varepsilon_t}} \leq 2\|v^2(0) - v^1(0)\|_{C^0_\star} \quad \text{with } \varepsilon_t := C \int_0^t V(\tau) d\tau, \quad \text{whenever } \varepsilon_t \leq \frac{1}{2}.$$

As  $v^2(0) - v^1(0) = 0$ , we get uniqueness on  $[0, T_0]$  with  $T_0 = \sup\{t \in [0, T], C \int_0^t V(\tau) d\tau \leq \frac{1}{2}\}$ . Because  $V \in L^1([0, T])$ , the argument may be repeated so that uniqueness holds on the whole interval  $[0, T]$ .

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