



Mathematical Analysis

Convergence of trigonometric series with general monotone coefficients

Mikhail Dyachenko ^a, Sergey Tikhonov ^b

^a *Moscow State University, Vorobevy Gory, 117234 Moscow, Russia*

^b *Scuola Normale Superiore, Pisa, Piazza dei Cavalieri 7, 56126 Pisa, Italy*

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Abstract

In this Note we study the convergence results for trigonometric series in L_p -spaces on one-dimensional and n -dimension torus. The sufficient conditions for these results to hold as well as criteria are written for the series with general monotone coefficients. The Hardy–Littlewood type theorem is obtained for multiple series. Several corollaries, in particular, u -convergence are presented. *To cite this article: M. Dyachenko, S. Tikhonov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

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Résumé

Convergence de séries trigonométriques à coefficients généraux monotones. Dans cette Note on établit des résultats de convergence de séries trigonométriques dans des espaces L_p unidimensionnels et sur le tore à n -dimensions. Des conditions suffisantes et des critères de validité sont établis pour des séries à coefficients généraux monotones. Un théorème de type Hardy–Littlewood est démontré dans le cas de séries multidimensionnelles. Des corollaires sont énoncés, en particulier dans le cas des séries multidimensionnelles. *Pour citer cet article : M. Dyachenko, S. Tikhonov, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

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1. One-dimensional trigonometric series

We consider the series

$$\sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

and

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{2}$$

E-mail addresses: dyach@mail.ru (M. Dyachenko), s.tikhonov@sns.it (S. Tikhonov).

where $\{a_n\}_{n=1}^{\infty}$ is a null sequence of complex numbers. We define by $f(x)$ and $g(x)$ the sums of the series (1) and (2) respectively at the points where the series converge.

First we note that the condition $\{a_n\} \in M$, i.e., $a_n \downarrow$, implies [14, V.1, p. 182] convergence of series (1) and (2) for all x , except possibly $x = 2\pi k$, $k \in \mathbf{Z}$, in the case of (1). The L_{∞} , L_1 , and L_p -convergence criteria for series (1) and (2) with decreasing coefficients are presented in the following theorems.

Theorem A. ([14, V.1, p. 183]) *If $\{a_n\} \in M$, then a necessary and sufficient condition for the uniform convergence of series (2) (or (2) a Fourier series of a continuous function) is the condition $\lim_{n \rightarrow \infty} na_n = 0$.*

Theorem B. [12] *Let $\{a_n\} \in M$. Let (1) be the Fourier series of a function $f(x) \in L_1$, and $S_n(f, x)$ be its n -partial sum. Then $\lim_{n \rightarrow \infty} \|f(\cdot) - S_n(f, \cdot)\|_1 = 0 \Leftrightarrow \lim_{n \rightarrow \infty} a_n \ln n = 0$. The same results hold for series (2) and $\sum_{n=-\infty}^{\infty} a_n e^{inx}$.*

Theorem C. ([14, V.2, XII, §6]) *Let $\{a_n\} \in M$, and let $1 < p < \infty$.*

Then f (or g) $\in L_p \Leftrightarrow \sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$.

Further, these results were generalized in many cases, particularly in consideration of some generalization of the M class. One can see that

$$M \subsetneq QM \cup RBVS \subsetneq ORVQM \cup RBVS,$$

where QM is a class of quasi monotone sequences, i.e. $QM = \{a: \exists \tau > 0 \text{ s.t. } n^{-\tau} a_n \downarrow\}$, $ORVQM$ is a class of O -regularly varying quasi monotone sequences [10], i.e.,

$$ORVQM = \{a: \exists \{\lambda_n\} \uparrow, \lambda_{2n} \leq C \lambda_n \text{ such that } \{\lambda_n^{-1} a_n\} \downarrow\}$$

and $RBVS = \{a: \sum_{v=n}^{\infty} |a_v - a_{v+1}| \leq C |a_n|\}$ (see [8]).

For any of this classes one can prove corresponding version of the convergence results (see [1,8,10–13] and reference there). We introduce the following concept:

Definition. Let $\beta = \{\beta_n\}_{n=1}^{\infty}$ be a non-negative sequence. The sequence of complex numbers $a = \{a_n\}_{n=1}^{\infty}$ is said to be β -general monotone, or $a \in GM(\beta)$, if the relation,

$$|a_n| + \sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C \beta_n,$$

holds for all integer n , where the constant C is independent of n [13].

In the case of $\beta_n := |a_n|$ we denote this class as GM . We note that $ORV \cup RBVS \subsetneq GM$ and $a \in GM$ iff $|a_v| \leq C |a_n|$ for $n \leq v \leq 2n$, and

$$\sum_{k=n}^N |\Delta a_k| \leq C \left(|a_n| + \sum_{k=n+1}^N \frac{|a_k|}{k} \right) \text{ for any } n < N. \quad (3)$$

We present the following analogues of Theorems A–C (see also [5,7], and [13]):

Theorem 1. (A) *Let $a \in GM(\beta)$ and $\sum_{n=1}^{\infty} \beta_{2^n} < \infty$. Then series (1) and (2) converge for all x except possibly $x = 2\pi k$, $k \in \mathbf{Z}$, in the case of (1), and converge uniformly on any interval $[\varepsilon, 2\pi - \varepsilon]$, where $0 < \varepsilon < \pi$. Moreover, if $n\beta_n = o(1)$ as $n \rightarrow \infty$, then firstly series (1) converges uniformly on $[0, 2\pi]$ iff $\sum_n a_n$ converges and secondly series (2) converges uniformly on $[0, 2\pi]$.*

(B) *If a positive sequence $a \in GM$, then series (2) converges uniformly on $[0, 2\pi]$ iff $\lim_{n \rightarrow \infty} na_n = 0$.*

Theorem 2. *Let (1) be the Fourier series of a function $f(x) \in L_1$.*

(A) *Suppose $a \in GM(\beta)$; then $\lim_{n \rightarrow \infty} \beta_n \ln n = 0 \Rightarrow \lim_{n \rightarrow \infty} \|f(\cdot) - S_n(f, \cdot)\|_1 = 0$.*

(B) *Suppose $a \in GM$; then $\lim_{n \rightarrow \infty} \|f(\cdot) - S_n(f, \cdot)\|_1 = 0 \Leftrightarrow \lim_{n \rightarrow \infty} |a_n| \ln n = 0$.*

The same results hold for series (2) and $\sum_{n=-\infty}^{\infty} a_n e^{inx}$.

Theorem 3. Let $1 < p < \infty$.

(A) Suppose $a \in GM(\beta)$; then $\sum_{n=1}^{\infty} \beta_n^p n^{p-2} < \infty \Rightarrow f(\text{or } g) \in L_p$.

(B) If $a = \{a_n\}$ is a positive sequence satisfying (3); then $f(\text{or } g) \in L_p \Leftrightarrow \sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$.

From these results we have the following corollary. If $a \in GM(\beta)$, where $\beta_n = n^{-\gamma}$, then the condition $\gamma > 1 - 1/p$ is sufficient for L_p -convergence of series (1) for $1 \leq p < \infty$ and of series (2) for $1 \leq p \leq \infty$. The next theorem is the result on convergence almost everywhere:

Theorem 4. Let $a \in GM(\beta)$ for $\beta_n = n^\gamma$, $\gamma \in [0, \frac{1}{2})$ and let $a_n = O(n^{-\delta})$, $\delta \in (\gamma, \frac{1}{2}]$, then series (1) and (2) converge almost everywhere.

Remark 5. The previous theorem is sharp in the following sense: (a) one cannot have convergence everywhere, (b) one cannot have convergence almost everywhere if $\delta = \gamma$. Actually, in the case of $\delta = \gamma$ we can have divergence almost everywhere. We also note that if $\delta > \frac{1}{2}$, convergence almost everywhere follows immediately from the Carleson’s theorem.

2. Multiple trigonometric series

In this section we study the n -dimension version of Hardy–Littlewood theorem for the following series ($N = \{1, 2, \dots, n\}$, $B \subseteq N$)

$$\sum_{\mathbf{m}=1}^{\infty} a_{\mathbf{m}} \prod_{j \in B} \cos m_j x_j \prod_{j \in N \setminus B} \sin m_j x_j. \tag{4}$$

We will assume that a_{m_1, \dots, m_n} such that $a_{\mathbf{m}} \rightarrow 0$ as $|\mathbf{m}| \equiv \sum_{j=1}^n m_j \rightarrow \infty$. We will need the following:

Definition. Let $\beta = \{\beta_{\mathbf{m}}\}$ be a non-negative sequence. We say that a sequence $a = \{a_{\mathbf{m}}\}$ satisfies $GM^n(\beta)$ -condition [5] if $\sum_{\mathbf{m}=\mathbf{k}}^{\infty} |\Delta^{1, \dots, 1} a_{\mathbf{m}}| \leq C^* \beta_{\mathbf{k}}$, where the operator $\Delta^{1, \dots, 1}$ is defined as follows: $\Delta^{1, \dots, 1} \equiv \prod_{j=1}^n \Delta^j$ and $\Delta^j a_{\mathbf{m}} = a_{\mathbf{m}} - a_{m_1, \dots, m_{j-1}, m_j+1, m_{j+1}, \dots, m_n}$. We know that in this case series (4) converges in the Pringsheim’s sense (over rectangles) everywhere on $(0, 2\pi)^2$ to a function f . Further, if

$$\begin{aligned} \beta_{\mathbf{k}} := & |a_{\mathbf{k}}| + \sum_{i=1}^n \sum_{m_i=k_i+1}^{\infty} \frac{|a_{k_1, \dots, k_{i-1}, m_i, k_{i+1}, \dots, k_n}|}{m_i} \\ & + \sum_{1 \leq i < j \leq n} \sum_{m_i=k_i+1}^{\infty} \sum_{m_j=k_j+1}^{\infty} \frac{|a_{k_1, \dots, m_i, \dots, m_j, \dots, k_n}|}{m_i m_j} + \dots + \sum_{\mathbf{m}=\mathbf{k}+1}^{\infty} \frac{|a_{m_1, \dots, m_n}|}{m_1 \cdots m_n} \end{aligned} \tag{5}$$

then we denote $a \in GM^n$. Note that this is the n -dimensional analogue of inequality (3).

Theorem 6. Let $1 < p < \infty$, $n \geq 1$.

(A) If $a \in GM^n(\beta)$, where the sequence β satisfies $I(\beta) := (\sum_{\mathbf{m}=1}^{\infty} \beta_{\mathbf{m}}^p (\prod_{j=1}^n m_j)^{p-2})^{1/p} < \infty$. Then the sum of series (4) is in L_p and $\|f(x)\|_p \leq C(C^*, p)I(\beta)$.

(B) If a non-negative sequence $a \in GM^n$, then $f \in L_p[0, 2\pi]^n$ iff $\sum_{\mathbf{m}=1}^{\infty} a_{\mathbf{m}}^p (\prod_{j=1}^n m_j)^{p-2} < \infty$.

Clearly, if $\Delta^{1, \dots, 1} a_{\mathbf{m}} \geq 0$, then $a \in GM^n$. In this case the part (B) was proved in [9]. Before presenting some corollaries we give two definitions.

Definition. Let $U \subset \mathbf{Z}^n$. Then we say that $U \in A$ if $\mathbf{k} \in U$ implies $\prod_{j=1}^n [-|k_j|, |k_j|] \cap \mathbf{Z}^n \subseteq U$. We also say that the numerical series $\sum_{\mathbf{m} \in \mathbf{Z}^n} c_{\mathbf{m}}$ u -converges to a number α , if for any $\varepsilon > 0$, there exists a number M such that for every $U \in A$ for which $\{\mathbf{m} \in \mathbf{Z}^n : |\mathbf{m}| \leq M\} \subseteq U$, we have $|\sum_{\mathbf{m} \in U} c_{\mathbf{m}} - \alpha| < \varepsilon$. Similarly, we define u -convergence in L_p . We note that u -convergence implies convergence over rectangles, over spheres, over hyperbolic crosses, etc (see for details [2–4] and reference there).

Corollary 6.1. Under the conditions of Theorem 6(A), series (4) u -converges in L_p for $\frac{2n}{n+1} < p < \infty$ and it is not true for $p \leq \frac{2n}{n+1}$.

Corollary 6.2. Let $1 < p < \infty$, and let $a \in GM^n$, $n \geq 1$. Then the series $(N = \{1, 2, \dots, n\}, B \subseteq N)$

$$\sum_{\mathbf{m}=1}^{\infty} A_{\mathbf{m}} \prod_{j \in B} \cos m_j x_j \prod_{j \in N \setminus B} \sin m_j x_j,$$

where $A_{\mathbf{m}} := \frac{1}{m_1 \dots m_n} \sum_{\mathbf{k}=1}^{\mathbf{m}} a_{\mathbf{k}}$, is the Fourier series a function $Hf \in L_p$ if and only if $f \in L_p$, $f \sim (4)$. Moreover, $\|f\|_p \asymp \|Hf\|_p \asymp (\sum_{\mathbf{m}=1}^{\infty} a_{\mathbf{m}}^p (\prod_{j=1}^n m_j)^{p-2})^{1/p}$.

Note that the part $\|Hf\|_p \leq C\|f\|_p$ for $n = 1$ is well-known Hardy theorem [6].

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