Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 345 (2007) 303–306
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Number Theory

Formal groups, Bernoulli-type polynomials and L-series

Piergiulio Tempesta

Scuola Normale Superiore, Centro di ricerca matematica Ennio de Giorgi, Piazza dei Cavalieri, 3, 56126 Pisa, Italy

Received 5 September 2006; accepted after revision 4 May 2007

Available online 4 September 2007

Presented by Jean-Pierre Serre

Abstract

A new construction relating formal groups, a class of Appell polynomials called the universal Bernoulli polynomials and a family of Dirichlet L-series is proposed. Universal Bernoulli χ-numbers as well as generalized Riemann–Hurwitz zeta functions are introduced. ***To cite this article: P. Tempesta, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
© 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Groupes formels, polynômes de type Bernoulli et séries L. On propose une nouvelle construction qui relie les groupes formels à une classe de polynomes de Appell qu' on appelle polynômes de Bernoulli universels et à une famille de séries de Dirichlet. On introduit aussi les nombres de Bernoulli universels liés à un caractère de Dirichlet χ et une généralisation des fonctions de Riemann–Hurwitz. ***Pour citer cet article : P. Tempesta, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***
© 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

The aim of this Note is to present the following construction: we associate with formal group laws new sequences of Bernoulli-type polynomials and Dirichlet L-series, in such a way that the correspondence between the classical Bernoulli polynomials and the Riemann zeta function is generalized.

As is well known, formal groups are relevant in many branches of mathematics, especially in the theory of elliptic curves [8], in algebraic topology [2] and analytic number theory [5]. We propose the following definition:

Definition 1.1. Let us consider the formal group logarithm, defined over the polynomial ring $\mathbb{Q}[c_1, c_2, \ldots]$

$$F(s) = s + c_1 \frac{s^2}{2} + c_2 \frac{s^3}{3} + \cdots. \tag{1}$$

Let $G(t)$ be the associated formal group exponential:

$$G(t) = t - c_1 \frac{t^2}{2!} + \left(3c_1^2 - 2c_2\right)\frac{t^3}{6!} + \cdots \tag{2}$$

E-mail address: p.tempesta@sns.it.

1631-073X/$ – see front matter © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2007.05.016

so that $F(G(t)) = t$. The universal higher-order Bernoulli polynomials $B_{k,a}^G(x, c_1, c_2, \ldots) \equiv B_{k,a}^G(x)$ are defined by

$$\left(\frac{t}{G(t)}\right)^a e^{xt} = \sum_{k\geqslant 0} B_{k,a}^G(x)\frac{t^k}{k!}, \quad x, a \in \mathbb{R}. \tag{3}$$

When $a = 1$, and $c_i = (-1)^i$, then $F(s) = \log(1 + s)$, $G(t) = e^t - 1$, and the universal Bernoulli polynomials and numbers reduce to the standard ones. In this Note, only the case $a = 1$ will be considered. By construction, the numbers $B_{k,1}^G(0) \in \mathbb{Q}[c_1, \ldots, c_k]$ coincide with the *universal Bernoulli numbers* introduced by Clarke in [4]. The name comes from the fact that $G(F(s_1) + F(s_2))$ is the universal formal group [5], which is defined over the Lazard ring L, i.e. the subring of $\mathbb{Q}[c_1, c_2, \ldots]$ generated by the coefficients of the power series $G(F(s_1) + F(s_2))$. The role of the Lazard group in algebraic topology has been clarified in [2]. For sake of simplicity, we put $B_{k,1}^G(x) \equiv B_k^G(x)$ and $B_k^G(0) \equiv B_k^G$. The classical Bernoulli numbers are fundamental in many branches of mathematics, like algebraic number theory and combinatorics, and have also applications in the Hirzebruch signature theorem, the computation of Todd characteristic classes and Gromov–Witten invariants. The universal Bernoulli numbers play as well an important role, in particular in complex cobordism theory (see e.g. [3], and [6]), where the coefficients c_n are identified with the cobordism classes of $\mathbb{C}P^n$. They also obey generalizations of the celebrated Kummer and Clausen–von Staudt congruences [1]. Many properties of the polynomials (3) are discussed in the paper [9]. The Appell property is expressed by

$$B_{n,a}^G(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_{k,a}^G(y)x^{n-k}. \tag{4}$$

In [9], it is shown that interesting realizations of the class of polynomials (3) can be constructed using the finite operator theory, introduced by G.C. Rota [7]. The formal exponential $G(t)$ is chosen to be the representative of a difference delta operator. This means that $G(t)$ is a Laurent polynomial in e^t (that we choose to have rational coefficients), obeying two more constraints:

$$G(t) = \sum_{k=l}^{m} a_k e^{kt}, \quad l, m \in \mathbb{Z},\ l < m, \tag{5}$$

with the two constraints

$$\sum_{k=l}^{m} a_k = 0, \qquad \sum_{k=l}^{m} k a_k = 1.$$

Let χ be a nontrivial Dirichlet character. It is also natural to generalize the Bernoulli numbers $B_{n,\chi}$.

Definition 1.2. Let χ be a nontrivial Dirichlet character of conductor N. The universal Bernoulli χ-numbers $B_{k,\chi}^G$ associated with the formal group G are defined by

$$N^{k-1} \sum_{b=1}^{N} \chi(b) B_k^G\left(\frac{b}{N}\right) = B_{k,\chi}^G. \tag{6}$$

Taking into account the summation formula for the universal Bernoulli polynomials, valid since they represent an Appell sequence for any choice of G, we obtain another equivalent representation for the numbers $B_{k,\chi}^G$:

$$B_{k,\chi}^G = \sum_{b=1}^{N} \chi(b) \sum_{j=0}^{k} \binom{k}{j} B_j^G b^{k-j} N^{j-1}. \tag{7}$$

2. L-series and formal groups

In this section, we will associate certain L-series with the class of polynomials (3). The Riemann zeta function is the most elementary example of the construction we propose. Let $\Gamma(s)=\int_0^\infty \mathrm{e}^{-t}t^{s-1}\,\mathrm{d}t$ be the Γ Euler function. As is well known, if $\operatorname{Re}s>1$,

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{1}{\mathrm{e}^t-1}t^{s-1}\,\mathrm{d}t.$$

More generally, by imposing suitable analytic constraints on the considered formal group exponentials, we can construct related classes of L-series. Precisely the following result holds:

Theorem 2.1. *Let $G(t)$ be a formal group exponential of the form* (2)*, such that $1/G(t)$ is a C^∞ function over $\mathbb{R}_+$, rapidly decreasing at infinity.*

(i) *The function*

$$L(G,s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{1}{G(t)}t^{s-1}\,\mathrm{d}t, \tag{8}$$

defined for $\operatorname{Re}s>1$ *admits an holomorphic continuation to the whole $\mathbb{C}$ and, for every $k\in\mathbb{N}$, we have*

$$L(G,-k)=(-1)^k\frac{B^G_{k+1}}{k+1}\in\mathbb{Q}[c_1,c_2,\ldots]. \tag{9}$$

(ii) *If $G(t)$ is also of the form* (5)*, for* $\operatorname{Re}s>1$ *the function $L(G,s)$ can be represented in terms of a Dirichlet series*

$$L(G,s)=\sum_{n=1}^\infty\frac{a_n}{n^s}, \tag{10}$$

where the coefficients a_n are obtained as the coefficients of the formal expansion

$$\frac{1}{G(t)}=\sum_{n=1}^\infty a_n\,\mathrm{e}^{-nt}. \tag{11}$$

Assuming that $G(t)\geqslant \mathrm{e}^t-1$, the series $L(G,s)$ is absolutely convergent for $\operatorname{Re}s>1$*, and*

$$\left|\sum_{n=1}^\infty\frac{a_n}{n^s}\right|\leqslant\sum_{n=1}^\infty\frac{1}{n^{\operatorname{Re}s}}. \tag{12}$$

Remark 2.1. Note that the functional relation $L(G,s)=L(G',s+1)$ holds.

Remark 2.2. A construction of L-series based on formal group laws is well known in the literature [5]. One of the advantages of the simple construction proposed here is that the associated L-functions are directly related to the universal Bernoulli numbers, whose algebraic and combinatorial properties are particularly rich.

3. Generalized Riemann–Hurwitz zeta functions

The previous ideas can be used to introduce a generalized Riemann–Hurwitz zeta function and to connect it in a natural way with the generalized Bernoulli polynomials (3).

The Riemann–Hurwitz zeta function is defined by the series

$$\zeta(s,v)=\sum_{n=0}^\infty(n+v)^{-s}, \tag{13}$$

for $v > 0$. As is well known, this series converges absolutely for $\operatorname{Re} s > 1$ and it extends to a meromorphic function in $\mathbb{C}$. The Riemann–Hurwitz zeta function for $k \in \mathbb{N}$ takes the special values

$$\zeta(-k, v) = -\frac{B_{k+1}(v)}{k+1}, \tag{14}$$

where $B_k(x)$ is the k-th classical Bernoulli polynomial. According to the previous scheme, we propose the following generalization of $\zeta(s, v)$:

Definition 3.1. Let $G(t)$ be a formal group exponential, such that $1/G(t)$ is a C^∞ function over $\mathbb{R}_+$, rapidly decreasing at infinity. The generalized Riemann–Hurwitz zeta function associated with G is the function $\zeta(G, s, v)$, defined for $\operatorname{Re} s > 1$ by

$$\zeta(G, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{e}^{x(1-v)}}{G(x)} x^{s-1}\, \mathrm{d}x. \tag{15}$$

Corollary 3.2. *The following property holds*:

$$\zeta(G, -k, v) = -\frac{B^G_{k+1}(v)}{k+1}, \tag{16}$$

where $B^G_k(x)$ is the k-th Bernoulli-type polynomial associated with the considered formal group.

Acknowledgements

I wish to thank professors B. Dubrovin, R.A. Leo, Don Zagier and U. Zannier for useful discussions and a careful reading of the manuscript.

References

[1] A. Adelberg, Universal higher order Bernoulli numbers and Kummer and related congruences, J. Number Theory 84 (2000) 119–135.
[2] V.M. Bukhshtaber, A.S. Mishchenko, S.P. Novikov, Formal groups and their role in the apparatus of algebraic topology, Uspekhi Mat. Nauk 26 (2) (1971) 161.
[3] A. Baker, Combinatorial and arithmetic identities based on formal group laws, in: Lecture Notes in Math., vol. 1298, Springer, 1987, pp. 17–34.
[4] F. Clarke, The universal von Staudt theorems, Trans. Amer. Math. Soc. 315 (1989) 591–603.
[5] M. Hazewinkel, Formal Groups and Applications, Academic Press, 1978.
[6] N. Ray, Stirling and Bernoulli numbers for complex oriented homology theory, in: G. Carlsson, R.L. Cohen, H.R. Miller, D.C. Ravenel (Eds.), Algebraic Topology, in: Lecture Notes in Math., vol. 1370, Springer-Verlag, 1986, pp. 362–373.
[7] G.C. Rota, Finite Operator Calculus, Academic Press, New York, 1975.
[8] J.-P. Serre, Courbes elliptiques et groupes formels, Annuaire du Collège de France (1966), 49–58 (Oeuvres, vol. II, 71, 315–324).
[9] P. Tempesta, New Appell sequences of polynomials of Bernoulli and Euler type, J. Math. Anal. Appl. (2007), in press.